D-BAR SPARK THEORY AND DELIGNE COHOMOLOGY

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ABSTRACT. We study the Harvey-Lawson spark characters of level p on complex manifolds. Presenting Deligne cohomology classes by sparks of level p, we give an explicit analytic product formula for Deligne cohomology. We also define refined Chern classes in Deligne cohomology for holomorphic vector bundles over complex manifolds. Applications to algebraic cycles are given. A Bott-type vanishing theorem in Deligne cohomology for holomorphic foliations is established. A general construction of Nadel-type invariants is given together with a new proof of Nadel's conjecture.

Contents

1. INTRODUCTION

In 1970's Cheeger and Simons [CS] introduced the ring of differential characters $\hat{H}^*(X)$ associated to a smooth manifold X. They generalized the Chern-Weil homomorphism and obtained a refinement of the theory of characteristic classes and characteristic forms with applications to conformal geometry, foliation theory and more. In 2003, Harvey, Lawson and Zweck [HLZ] gave a new description of differential characters from a de Rham-Federer viewpoint and established the ring of de Rham-Federer spark classes which is isomorphic to the Cheeger-Simons differential characters. Later, Harvey and Lawson expanded their approach to the theory of differential characters in [HL2]. They invented a homological apparatus to study differential characters systematically and introduced many different presentations of differential characters. Central to their theory are *spark complexes*, *sparks* and *rings of spark characters* which are analogues of cochain complexes, cocycles and cohomology rings in the usual cohomology theory. Roughly speaking, a spark complex is a triple of cochain complexes, two of which are contained in the main one with trivial intersection. A spark is an element in the main complex such that its differential can be represented (uniquely) as the sum of elements from the other two complexes. An equivalence relation among sparks is

introduced and the group of spark classes is established. It turns out that spark complexes are abundant in geometry, topology and physics. A more striking fact is that the classical secondary invariants like Cheeger-Simons differential characters can be realized as the groups of spark classes associated to many different spark complexes. Therefore, these spark complexes give many different presentations of differential characters just as there are many different presentations of cohomology. One basic example is the *de Rham-Federer spark complex* introduced in [HLZ]. This is the complex whose objects are currents whose exterior differentials can be decomposed into smooth forms and rectifiable currents. Among many other examples studied in [HL2], the *smooth hyperspark complex* is closely related to n-gerbes with connections in physics, the *Cheeger-Simons spark complex* is the closest one to the Cheeger-Simons definition. Furthermore, all spark complexes appearing above are compatible, which implies the groups of spark classes associated to them are all isomorphic. We call these groups the Harvey-Lawson spark characters of a smooth manifold collectively and denote them by $\hat{\mathbf{H}}^*(X)$.

In [HLZ], a ring structure was constructed on the group of de Rham-Federer spark classes $\hat{\mathbf{H}}^*_{spark}(X)$ using a transversality theorem for currents. A *de Rham-Federer spark* is a current $\mathbf{a}$ satisfying the *spark equation* $d\mathbf{a} = \mathbf{e} - \mathbf{r}$ for some smooth differential form $\mathbf{e}$ and rectifiable current $\mathbf{r}$. If we have two spark classes α and β with representatives $\mathbf{a}$ and $\mathbf{b}$ satisfying spark equations $d\mathbf{a} = \mathbf{e} - \mathbf{r}$ and $d\mathbf{b} = \mathbf{f} - \mathbf{s}$, we may define the product $\alpha \cdot \beta$ as the spark class represented by $\mathbf{a} \wedge \mathbf{f} + (-1)^{\deg \mathbf{a}+1} \mathbf{r} \wedge \mathbf{b}$ which satisfies the spark equation $d(\mathbf{a} \wedge \mathbf{f} + (-1)^{\deg \mathbf{a}+1} \mathbf{r} \wedge \mathbf{b}) = \mathbf{e} \wedge \mathbf{f} - \mathbf{r} \wedge \mathbf{s}$. But we have to worry about the well-definedness of the wedge product of two currents. It was shown in [HLZ] by geometric measure theory that there always exist good representatives such that all wedge products in the formula make sense and the spark class of the product is independent of the choices of representatives. Therefore, a ring structure on $\hat{\mathbf{H}}^*_{spark}(X)$ is established.
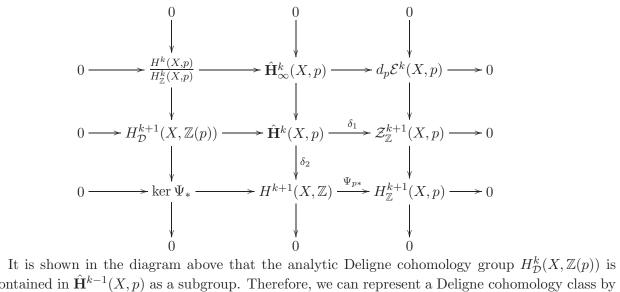
In [H1], we focused on the smooth hyperspark complex and gave a construction of the multiplication in the group of smooth hyperspark classes $\hat{\mathbf{H}}^*_{smooth}(X)$. Fix a good cover $\mathcal{U}$ of X, a *smooth hyperspark* is an element in the Čech-de Rham double complex

$$\mathbf{a} \in \bigoplus_{p+q=k} C^p(\mathcal{U}, \mathcal{E}^q)$$

with the spark equation $D\mathbf{a} = \mathbf{e} - \mathbf{r}$ where $\mathbf{e} \in \mathcal{E}^{k+1}(X) \subset C^0(\mathcal{U}, \mathcal{E}^{k+1})$ and $\mathbf{r} \in C^{k+1}(\mathcal{U}, \mathbb{Z})$. We introduced a cup product $\cup$ on the Čech-de Rham double complex and defined the product of two sparks $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cup \mathbf{f} + (-1)^{\deg \mathbf{a}+1} \mathbf{r} \cup \mathbf{b}$ for sparks $\mathbf{a}$ and $\mathbf{b}$ satisfying spark equations $D\mathbf{a} = \mathbf{e} - \mathbf{r}$ and $D\mathbf{b} = \mathbf{f} - \mathbf{s}$. This product induces a product in the group of smooth hyperspark classes $\hat{\mathbf{H}}^*_{smooth}(X)$. Moreover, these two ring structures were shown to be compatible with the group isomorphism $\hat{\mathbf{H}}^*_{spark}(X) \cong \hat{\mathbf{H}}^*_{smooth}(X)$.

In a recent paper [HL3], Harvey and Lawson developed a theory on $\bar{\partial}$-analogue of differential characters for complex manifolds and introduced the Harvey-Lawson spark characters of level p. While Harvey and Lawson concentrated on the spark characters of level 1 in [HL3], we generalize their theory to level p for any positive integer p in the first half of this paper. To study the spark characters of level p, we introduce the *Dolbeault-Federer spark complex* of level p, the *Čech-Dolbeault spark complex* of level p and the *Čech-Dolbeault hyperspark complex* of level p associated to a complex manifold X. These spark complexes are truncated versions of the *de Rham-Federer spark complex*, the *smooth hyperspark complex* and the *hyperspark complex* which were first introduced in [HL2] and [HL3]. It is not surprising that the groups associated to these truncated spark complexes are isomorphic to each other. We denote these groups by $\hat{\mathbf{H}}^*(X, p)$ collectively and call them the Harvey-Lawson spark characters of level p. Furthermore, there is a group epimorphism $\Pi_p : \hat{\mathbf{H}}^*(X) \to \hat{\mathbf{H}}^*(X, p)$, whose kernel is an ideal. Hence we establish the ring structure by identifying $\hat{\mathbf{H}}^*(X, p)$ as the quotient ring of spark characters $\hat{\mathbf{H}}^*(X)$.

One important application of the theory of spark characters is to study analytic Deligne cohomology. We have the following fundamental 3×3 grid for $\hat{\mathbf{H}}^*(X, p)$.

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{H^k(X,p)}{H^k_{\mathbb{Z}}(X,p)} & \longrightarrow & \hat{\mathbf{H}}^k_\infty(X,p) & \longrightarrow & d_p\mathcal{E}^k(X,p) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^{k+1}_{\mathcal{D}}(X,\mathbb{Z}(p)) & \longrightarrow & \hat{\mathbf{H}}^k(X,p) & \overset{\delta_1}{\longrightarrow} & \mathcal{Z}^{k+1}_{\mathbb{Z}}(X,p) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\scriptstyle\delta_2} & & \downarrow & & \\
0 & \longrightarrow & \ker \Psi_* & \longrightarrow & H^{k+1}(X,\mathbb{Z}) & \overset{\Psi_{p*}}{\longrightarrow} & H^{k+1}_{\mathbb{Z}}(X,p) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

It is shown in the diagram above that the analytic Deligne cohomology group $H^k_{\mathcal{D}}(X,\mathbb{Z}(p))$ is contained in $\hat{\mathbf{H}}^{k-1}(X,p)$ as a subgroup. Therefore, we can represent a Deligne cohomology class by a spark of level p. Lifting Deligne classes to spark classes in $\hat{\mathbf{H}}^*(X)$ and using the product formula for spark characters introduced earlier, we give an explicit product formula for analytic Deligne cohomology. We also show this product is the same as the product invented by Beilinson in [B]. Moreover, we can expand our theory to study higher operations, i.e. Massey products for Deligne cohomology, which was introduced by Deninger [De] in 1995. We shall study Massey products in spark characters, as well as Massey products in Deligne cohomology in [H2].

By the spark presentation of Deligne classes, it is transparent to see that every analytic subvariety of a complex manifold represents a Deligne class. As a direct application of our product formula for Deligne classes, we show that the intersection of two subvarieties represents the product of their Deligne classes if they intersect properly. In particular, when the setting is algebraic, we have a direct way to construct the cycle map $\psi : CH^*(X) \to H^{2*}_{\mathcal{D}}(X,\mathbb{Z}(*))$ via our theory.

Cheeger and Simons [CS] constructed Chern classes in differential characters for complex vector bundles with connections which refined the usual Chern classes. For holomorphic vector bundles over a complex manifold, we give a Chern-Weil-type construction for Chern classes in Deligne cohomology via Cheeger-Simons theory. To construct Chern classes in Deligne cohomology for a holomorphic vector bundle, we choose any connection compatible with the holomorphic structure, and project the kth Cheeger-Simons Chern class to $\hat{\mathbf{H}}^{2k-1}(X,k)$ whose image is actually in $H^{2k}_{\mathcal{D}}(X,\mathbb{Z}(k))$. We show that the image of kth Cheeger-Simons Chern in $H^{2k}_{\mathcal{D}}(X,\mathbb{Z}(k))$ is independent of the choice of connection and define it as our kth Chern class. The functorial property of Chern classes and the Whitney formula are shown as well. In [Z], Zucker indicated that the splitting principle works well in defining Chern classes in Deligne cohomology. In contrast to Zucker's method, our method is constructive since it is possible to explicitly construct representatives of Cheeger-Simons Chern classes via methods of Harvey-Lawson [HL1] or Brylinski-McLaughlin [BrM].

In 1969, Bott [Bo] constructed a family of connections on the normal bundle of any smooth foliation of a manifold and established the Bott vanishing theorem which says the characteristic classes of the normal bundle are trivial in all sufficiently high degrees. We prove an analogue of the Bott vanishing theorem for our refined Chern classes of the normal bundle of a holomorphic foliation.

In 1997, Nadel [N] introduced relative invariants for holomorphic vector bundles. Explicitly, for two holomorphic vector bundles E and F over a complex manifold X which are C^∞ isomorphic,

Nadel defined invariants $\mathscr{E}^k(E, F) \in H^{2k-1}(X, \mathcal{O})/H^{2k-1}(X, \mathbb{Z})$. He also conjectured that these invariants should coincide with a component of the Abel-Jacobi image of $k!(ch_k(E) - ch_k(F)) \in CH^k_{hom}(X)$ when the setting is algebraic. This conjecture was proved by Berthomieu [Be] by his relative K-theory. Since we define Chern classes and Chern characters in Deligne cohomology for holomorphic vector bundles, we can construct Nedel-type invariants $\hat{\mathscr{E}}^k(E, F)$ in intermediate Jacobians in more general setting (not necessarily algebraic). Moreover, we show that $\hat{\mathscr{E}}^k(E, F)$ is represented by a smooth $2k-1$-form whose $(0, 2k-1)$ component represents $\mathscr{E}^k(E, F) \in H^{2k-1}(X, \mathcal{O})/H^{2k-1}(X, \mathbb{Z})$. Therefore, Nadel's conjecture is proved in a more general context.

The organization of this paper is the following. First we recall the definition and basic properties of (generalized) spark complex invented by Harvey and Lawson [HL2] [HL3]. The main examples — the Dolbeault-Federer spark complex, the Čech-Dolbeault spark complex, the Čech-Dolbeault hyperspark complex — are introduced in Section 3-5. The ring structure and functoriality of spark characters associated to these spark complexes are established and the equivalence of them are also verified. Using product formula of spark characters introduced in [HLZ] [H1], we give an explicit product formula for analytic Deligne cohomology in Section 6. We apply spark theory to algebraic cycles and define the ring homomorphism from Chow ring to Deligne cohomology in Section 7. In Section 8, we define the refined Chern classes in Deligne cohomology for holomorphic vector bundles over complex manifolds. And in Section 9, we prove an analogue of the Bott vanishing theorem for holomorphic foliations in this context. In Section 10, we define Nadel-type invariants for holomorphic vector bundles in the intermediate Jacobians and prove Nadel's conjecture.

Acknowledgements. I am very grateful to my advisor H. Blaine Lawson for introducing this subject to me. I am also indebted to him for his encouragement and helpful discussions during the preparation of this paper. I would like to thank Zhiwei Yun and Li Li for useful discussions.

2. Generalized Spark Complexes

We follow [HL3] to give the definitions of a generalized spark complex and its associated group of spark classes which are generalizations of the spark complex and its associated group of spark classes appeared in [HL2] [H1]. When we mention spark complex in this paper, we mean this generalized spark complex defined below.

Definition 2.1. *A (generalized)* **homological spark complex**, *or* **spark complex** *for short, is a triple of cochain complexes (F^*, E^*, I^*) together with morphisms*

$$I^* \xrightarrow{\Psi} F^* \hookleftarrow E^*$$

such that

(1) $I^k \cap E^k = 0$ *for* $k > 0$, $F^k = E^k = I^k = 0$ *for* $k < 0$,
(2) $H^*(E^*) \cong H^*(F^*)$,
(3) $\Psi|_{I^0} : I^0 \to F^0$ *is injective.*

Definition 2.2. *In a given spark complex (F^*, E^*, I^*), a* **spark** *of degree k is a pair $(a, r) \in F^k \oplus I^{k+1}$ which satisfies the* **spark equations**

(1) $da = e - \Psi(r)$ *for some* $e \in E^{k+1}$,
(2) $dr = 0$.

Two sparks $(a, r), (a', r')$ of degree k are **equivalent** *if there exists a pair $(b, s) \in F^{k-1} \oplus I^k$ such that*

(1) $a - a' = db + \Psi(s)$,
(2) $r - r' = -ds$.

The set of equivalence classes is called the group of spark classes of degree k and denoted by $\hat{\mathbf{H}}^k(F^*, E^*, I^*)$ or $\hat{\mathbf{H}}^k$ for short. Let $[(a, r)]$ denote the equivalence class containing the spark (a, r).

Remark 2.3. *Harvey and Lawson introduced spark complexes in [HL2] where they required Ψ to be injective. In that case, $\mathbf{e}$ and $\mathbf{r}$ are uniquely determined by $\mathbf{a}$. Later, they generalized the original definition and defined the generalized spark complex in [HL3] where $\Psi : I^* \to F^*$ was not required to be injective. Hence, $\mathbf{r}$ is not determined uniquely by $\mathbf{a}$ and we have to remember $\mathbf{r}$ for a spark and denote a spark by $(\mathbf{a}, \mathbf{r})$.*

Also, in this paper, when we discuss spark complexes in which Ψ is injective, we may denote a spark only by $\mathbf{a}$ and omit $\mathbf{r}$.

We now derive the fundamental exact sequences associated to a spark complex (F^*, E^*, I^*). Let $Z^k(E^*) = \{e \in E^k : de = 0\}$ and set

$$Z_I^k(E^*) \equiv \{e \in Z^k(E^*) : [e] = \Psi_*([r]) \text{ for some } [r] \in H^k(I^*)\}$$

where $[e]$ denotes the class of e in $H^k(E^*) \cong H^k(F^*)$.

Lemma 2.4. *There exist well-defined surjective homomorphisms*

$$\delta_1 : \hat{\mathbf{H}}^k \to Z_I^{k+1}(E^*) \quad and \quad \delta_2 : \hat{\mathbf{H}}^k \to H^{k+1}(I^*)$$

given by

$$\delta_1([(a, r)]) = e \quad and \quad \delta_2([(a, r)]) = [r]$$

where $da = e - \Psi(r)$.

Proof. If (a', r') is equivalent to (a, r), i.e. $a - a' = db + \Psi(s)$ and $r - r' = -ds$, then we have $da' = e - \Psi(r + ds)$. So it is easy to see these maps are well-defined.

Consider $e \in Z_I^{k+1}(E^*)$, by definition, there exists $r \in I^{k+1}$ such that $e - \Psi(r)$ is exact in F^{k+1}, i.e. $\exists a \in F^k$ with $da = e - \Psi(r)$. So $\delta_1([(a, r)]) = e$. For $[r] \in H^{k+1}(I^*)$, $\Psi(r)$ also represents a class in $H^{k+1}(F^*) \cong H^{k+1}(E^*)$. Choosing a representative $e \in E^{k+1}$ of this class, we have $e - \Psi(r) = da$ for some $a \in F^k$, hence $\delta_2([(a, r)]) = [r]$. Both δ_1 and δ_2 are surjective. $\square$

Lemma 2.5. *Define $\hat{\mathbf{H}}_E^k \equiv \ker \delta_2$, then $\hat{\mathbf{H}}_E^k \cong E^k/Z_I^k(E^*)$.*

Proof. Let $\alpha \in \hat{\mathbf{H}}_E^k$ be represented by (a, r) with spark equations $da = e - \Psi(r)$ and $dr = 0$. Then we have $[r] = \delta_2(\alpha) = 0$, i.e. $r = -ds$ for some $s \in I^k$. So $d(a - \Psi(s)) = e$, by [HL2, Lemma 1.5] and the fact $H^*(F^*) \cong H^*(E^*)$, there exists $b \in F^{k-1}$ such that $a' \equiv a - \Psi(s) + db \in E^k$. Hence α can be represented by spark $(a', 0)$ with $a' \in E^k$. If $(a', 0)$ is equivalent to 0, then $a' = db' + \Psi(s')$ for some $b' \in F^{k-1}$ and $s' \in I^k$ with $ds' = 0$, i.e. $a' \in Z_I^k(E^*)$. $\square$

Remark 2.6. *From last proof, it is easy to see that $\hat{\mathbf{H}}_E^k$ is the space of spark classes that can be represented by sparks of type $(a, 0)$ where $a \in E^k$.*

Definition 2.7. *Associated to any spark complex (F^*, E^*, I^*) is the cone complex (G^*, D) defined by setting*

$$G^k \equiv F^k \oplus I^{k+1} \text{ with differential } D(a, r) = (da + \Psi(r), -dr).$$

Consider the homomorphism $\Psi_* : H^k(I^*) \to H^k(F^*) = H^k(E^*)$, and define

$$H_I^k(E^*) \equiv \text{Image}\{\Psi_*\} \quad and \quad Ker^k(I^*) \equiv \ker\{\Psi_*\}.$$

Proposition 2.8. [HL3] *There are two fundamental short exact sequences*

(1) $0 \longrightarrow H^k(G^*) \longrightarrow \hat{\mathbf{H}}^k \xrightarrow{\delta_1} Z_I^{k+1}(E^*) \longrightarrow 0$;

(2) $0 \longrightarrow \hat{\mathbf{H}}_E^k \longrightarrow \hat{\mathbf{H}}^k \xrightarrow{\delta_2} H^{k+1}(I^*) \longrightarrow 0$.

Moreover, associated to any spark complex (F^*, E^*, I^*) is the commutative diagram

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \dfrac{H^k(E^*)}{H_I^k(E^*)} & \longrightarrow & \hat{\mathbf{H}}_E^k & \longrightarrow & dE^k & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^k(G^*) & \longrightarrow & \hat{\mathbf{H}}^k & \xrightarrow{\ \delta_1\ } & Z_I^{k+1}(E^*) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\scriptstyle\delta_2} & & \downarrow & & \\
0 & \longrightarrow & Ker^{k+1}(I^*) & \longrightarrow & H^{k+1}(I^*) & \longrightarrow & H_I^{k+1}(E^*) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

whose rows and columns are exact.

Also, we can talk about quasi-isomorphism between two (generalized) spark complexes.

Definition 2.9. *Two spark complexes (F^*, E^*, I^*) and $(\bar{F}^*, \bar{E}^*, \bar{I}^*)$ are quasi-isomorphic if there exists a commutative diagram of morphisms*

$$
\begin{array}{ccccc}
I^* & \xrightarrow{\ \Psi\ } & F^* & \xleftarrow{\ i\ } & E^* \\
{\scriptstyle\psi}\downarrow & & {\scriptstyle i}\downarrow & & \| \\
\bar{I}^* & \xrightarrow{\ \bar{\Psi}\ } & \bar{F}^* & \xleftarrow{\ i\ } & \bar{E}^*
\end{array}
$$

inducing an isomorphism

$$
\psi^* : H^*(I^*) \xrightarrow{\ \cong\ } H^*(\bar{I}^*).
$$

Proposition 2.10. [HL3] *A quasi-isomorphism of spark complexes (F^*, E^*, I^*) and $(\bar{F}^*, \bar{E}^*, \bar{I}^*)$ induces an isomorphism*

$$
\hat{\mathbf{H}}^*(F^*, E^*, I^*) \cong \hat{\mathbf{H}}^*(\bar{F}^*, \bar{E}^*, \bar{I}^*)
$$

of the associated groups of spark classes. Moreover, it induces an isomorphism of the 3×3 grids associated to these two complexes.

3. Dolbeault-Federer Sparks of Level p

Let X be a complex manifold. Recall the **de Rham-Federer spark complex** [HLZ] associated to X is a spark complex (F^*, E^*, I^*) where

$$
F^* \equiv \mathcal{D}'^*(X), E^* \equiv \mathcal{E}^*(X), I^* \equiv \mathcal{IF}^*(X).
$$

Note that $\mathcal{E}^*$ and $\mathcal{D}'^*$ denote the sheaves complex-valued smooth forms and currents respectively. And $\mathcal{IF}^*$ is the sheaf of locally integrally flat currents on X. The associated group of spark classes is denoted by $\hat{\mathbf{H}}_{spark}^*(X)$, or $\hat{\mathbf{H}}^*(X)$ for short. In fact, $\hat{\mathbf{H}}_{spark}^*(X)$ is a ring and functorial with respect to smooth map between manifolds. We refer to [HL2] [HLZ] for details.

Now we introduce a new spark complex, the Dolbeault-Federer spark complex of level p, which is closely related to the de Rham-Federer spark complex.

For a complex manifold X, we can decompose the space of smooth k-forms by types:

$$
\mathcal{E}^k(X) \equiv \bigoplus_{r+s=k} \mathcal{E}^{r,s}(X).
$$

And similarly,
$$\mathcal{D}'^k(X) \equiv \bigoplus_{r+s=k} \mathcal{D}'^{r,s}(X).$$

Fix an integer $p > 0$ and consider the truncated complex $(\mathcal{D}'^*(X, p), d_p)$ with
$$\mathcal{D}'^k(X, p) \equiv \bigoplus_{r+s=k, r<p} \mathcal{D}'^{r,s}(X) \text{ and } d_p \equiv \pi_p \circ d$$

where $\pi_p : \mathcal{D}'^k(X) \to \mathcal{D}'^k(X, p)$ is the natural projection $\pi_p(a) = a^{0,k} + \dots + a^{p-1,k-p+1}$. Similarly, we can define $(\mathcal{E}^*(X, p), d_p)$.

Definition 3.1. *By the **Dolbeault-Federer spark complex of level p**, or more simply, the* $\bar{\mathbf{d}}$***-spark complex of level p** we mean the triple* (F_p^*, E_p^*, I_p^*)
$$F_p^* \equiv \mathcal{D}'^*(X, p), \quad E_p^* \equiv \mathcal{E}^*(X, p), \quad I_p^* \equiv \mathcal{IF}^*(X)$$

with maps
$$E_p^* \hookrightarrow F_p^* \quad and \quad \Psi_p : I_p^* \to F_p^*$$

where $\Psi_p = \pi_p \circ i$.

Remark 3.2. *The triple* $(F_p^*, E_p^*, I_p^*) \equiv (\mathcal{D}'^*(X, p), \mathcal{E}^*(X, p), \mathcal{IF}^*(X))$ *is a spark complex.*

Proof. First,
$$H^*(F_p^*) \cong H^*(E_p^*) \cong \mathbb{H}^*(\Omega^{*<p}) \equiv H^*(X, p).$$

where $\mathbb{H}^*(\Omega^{*<p}) \equiv H^*(X, p)$ denotes the hypercohomology of complex of sheaves
$$0 \to \Omega^0 \to \Omega^1 \to \Omega^2 \to \cdots \to \Omega^{p-1} \to 0$$

and Ω^k is the sheaf of holomorphic k-forms on X.

For the proof of $\Psi_p(I_p^k) \cap E_p^k = \{0\}$ for $k > 0$, we refer to [HL3, Appendix B]. $\square$

Definition 3.3. *A **Dolbeault-Federer spark of level p** of degree k, or a* $\bar{\mathbf{d}}$***-spark of level p** is a pair*
$$(a, r) \in \mathcal{D}'^k(X, p) \oplus \mathcal{IF}^{k+1}(X)$$

satisfying the spark equations
$$d_p a = e - \Psi_p(r) \quad and \quad dr = 0$$

for some $e \in \mathcal{E}^{k+1}(X, p)$.

*Two Dolbeault-Federer sparks of level p, (a, r) and (a', r') are **equivalent** if there exist $b \in \mathcal{D}'^{k-1}(X, p)$ and $s \in \mathcal{IF}^k(X)$ such that*
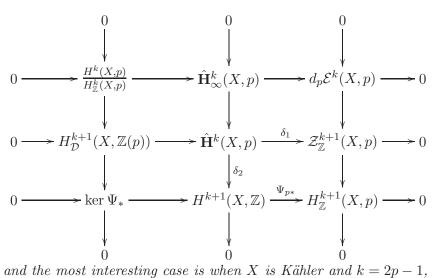$$a - a' = d_p b + \Psi_p(s) \quad and \quad r - r' = -ds.$$

The equivalence class determined by a spark (a, r) will be denoted by $[(a, r)]$, and the group of Dolbeault-Federer spark classes of level p of degree k will be denoted by $\hat{\mathbf{H}}_{spark}^k(X, p)$ or $\hat{\mathbf{H}}^k(X, p)$ for short.
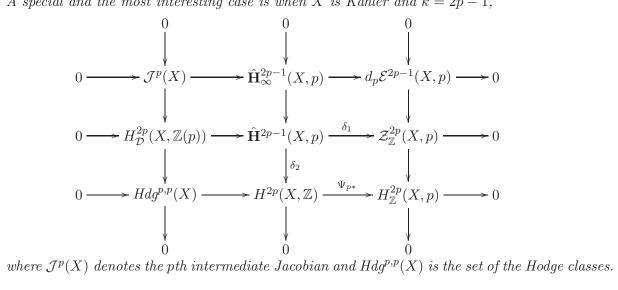
Applying Proposition 2.8, we have

Proposition 3.4. *Let $H_{\mathbb{Z}}^{k+1}(X, p)$ denote the image of map $\Psi_{p*} : H^{k+1}(X, \mathbb{Z}) \to H^{k+1}(X, p)$, and $\mathcal{Z}_{\mathbb{Z}}^{k+1}(X, p)$ denote the set of d_p-closed forms in $\mathcal{E}^{k+1}(X, p)$ which represent classes in $H_{\mathbb{Z}}^{k+1}(X, p)$. Let $\hat{\mathbf{H}}_{\infty}^k(X, p)$ denote the spark classes representable by smooth forms, and $H_{\mathcal{D}}^{k+1}(X, \mathbb{Z}(p))$ denote the Deligne cohomology group.*

The 3×3 diagram for $\hat{\mathbf{H}}^k(X, p)$ can be written as

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{H^k(X,p)}{H^k_{\mathbb{Z}}(X,p)} & \longrightarrow & \hat{\mathbf{H}}^k_\infty(X,p) & \longrightarrow & d_p\mathcal{E}^k(X,p) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^{k+1}_{\mathcal{D}}(X,\mathbb{Z}(p)) & \longrightarrow & \hat{\mathbf{H}}^k(X,p) & \overset{\delta_1}{\longrightarrow} & \mathcal{Z}^{k+1}_{\mathbb{Z}}(X,p) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\scriptstyle\delta_2} & & \downarrow & & \\
0 & \longrightarrow & \ker\Psi_* & \longrightarrow & H^{k+1}(X,\mathbb{Z}) & \overset{\Psi_{p*}}{\longrightarrow} & H^{k+1}_{\mathbb{Z}}(X,p) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

A special and the most interesting case is when X is Kähler and $k = 2p - 1$,

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{J}^p(X) & \longrightarrow & \hat{\mathbf{H}}^{2p-1}_\infty(X,p) & \longrightarrow & d_p\mathcal{E}^{2p-1}(X,p) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p)) & \longrightarrow & \hat{\mathbf{H}}^{2p-1}(X,p) & \overset{\delta_1}{\longrightarrow} & \mathcal{Z}^{2p}_{\mathbb{Z}}(X,p) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\scriptstyle\delta_2} & & \downarrow & & \\
0 & \longrightarrow & Hdg^{p,p}(X) & \longrightarrow & H^{2p}(X,\mathbb{Z}) & \overset{\Psi_{p*}}{\longrightarrow} & H^{2p}_{\mathbb{Z}}(X,p) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

where $\mathcal{J}^p(X)$ denotes the pth intermediate Jacobian and $Hdg^{p,p}(X)$ is the set of the Hodge classes.

Proof. The proof follows Proposition 2.8 directly. The only nontrivial part is why Deligne cohomology appears in the middle row. We postpone our proof to Section 6 where we study Deligne cohomology in detail. $\qquad\square$

Remark 3.5. *The $\bar{d}$-spark complex is a generalization of $\bar{\partial}$-spark complex in* [HL3] *which corresponds the special case $p = 1$.*

3.1. **Ring Structure.** We can establish the ring structure on $\hat{\mathbf{H}}^*(X,p)$ by identifying it as a quotient ring of $\hat{\mathbf{H}}^*(X)$.

Consider the following commutative diagram:

$$
\begin{array}{ccccc}
\mathcal{IF}^*(X) & \overset{i}{\longrightarrow} & \mathcal{D}'^*(X) & \overset{i}{\longleftarrow} & \mathcal{E}^*(X) \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle\pi_p} & & \downarrow{\scriptstyle\pi_p} \\
\mathcal{IF}^*(X) & \overset{\pi_p\circ i}{\longrightarrow} & \mathcal{D}'^*(X,p) & \overset{i}{\longleftarrow} & \mathcal{E}^*(X,p)
\end{array}
$$

which induces a group homomorphism $\Pi_p : \hat{\mathbf{H}}^*(X) \to \hat{\mathbf{H}}^*(X,p)$. Furthermore, we have

Theorem 3.6. *The morphism of spark complexes* $(\pi_p, \pi_p, id) : (F^*, E^*, I^*) \to (F_p^*, E_p^*, I_p^*)$ *induces a surjective group homomorphism*

$$\Pi_p : \hat{\mathbf{H}}^*(X) \to \hat{\mathbf{H}}^*(X, p)$$

whose kernel is an ideal. Hence, $\hat{\mathbf{H}}^*(X, p)$ *carries a ring structure.*

Proof. It's straightforward to see that the diagram above commutes and π_p commutes with differentials. Consequently, the induced map $(a, r) \mapsto (\pi_p(a), r)$ on sparks descends to a well defined homomorphism $\Pi_p : \hat{\mathbf{H}}^k(X) \to \hat{\mathbf{H}}^k(X, p)$ as claimed.

To prove the surjectivity, consider a spark $(A, r) \in F_p^k \oplus I^{k+1}$ with $dr = 0$ and $d_p A = e - \Psi_p(r)$ for some $e \in E_p^{k+1}$. We can choose some smooth form which represents same cohomology class with r in $H^{k+1}(F^*) \cong H^{k+1}(E^*)$, so there exist $a_0 \in F^k$, $e_0 \in E^{k+1}$ such that $da_0 = e_0 - r$. We have $\pi_p(da_0) = \pi_p(e_0 - r) \Rightarrow d_p(\pi_p a_0) = \pi_p e_0 - \Psi_p r$. Hence, $d_p(A - \pi_p a_0) = e - \pi_p e_0$ is a smooth form. It follows by [HL2, Lemma 1.5] that there exist $b \in F_p^{k-1}$ and $f \in E_p^k$ with $A - \pi_p a_0 = f + d_p b$. Set $a = a_0 + f + db$ and note that $da = da_0 + df + ddb = e_0 - r + df = (e_0 + df) - r$. Hence, (a, r) is a spark of degree k and $\pi_p a = \pi_p(a_0 + f + db) = \pi_p a_0 + f + d_p b = A$. So Π_p is surjective.

We need the following lemma to show the kernel is an ideal.

Lemma 3.7. *On* $\hat{\mathbf{H}}^k(X)$, *one has that* $\ker(\Pi_p) = \{\alpha \in \hat{\mathbf{H}}^k(X) : \exists (a, 0) \in \alpha \text{ where } a \text{ is smooth and } \pi_p(a) = 0\}$. *In particular,* $\ker(\Pi_p) \subset \hat{\mathbf{H}}_\infty^k(X)$.

Proof. One direction is clear. Suppose $\alpha \in \ker(\Pi_p)$ and choose any spark $(a, r) \in \alpha$. $\Pi_p(\alpha) = 0$ means that there exist $b \in F_p^{k-1}$ and $s \in I^k$ with

$$\begin{cases} \pi_p(a) = d_p b + \Psi_p(s) = \pi_p(db + s) \\ r = -ds \end{cases}$$

Replace (a, r) by $(\tilde{a}, 0) = (a - db - s, r + ds)$, note that $\pi_p(\tilde{a}) = \pi_p(a - db - s) = 0$.

In fact, we can choose $\tilde{a}$ to be smooth. $d\tilde{a} = da - ds = e - r - ds = e$ is a smooth form, it follows by [HL2, Lemma 1.5] and the fact $H^*(F^p \mathcal{D}'^*(X)) = H^*(F^p \mathcal{E}^*(X))$ that we can choose $\tilde{a}$ to be smooth. Note that $F^0 \mathcal{D}'^* \supset F^1 \mathcal{D}'^* \supset \cdots \supset F^p \mathcal{D}'^* \supset \cdots$ is the naive filtration. $\square$

By the product formula of $\hat{\mathbf{H}}^*(X)$ in [HLZ], it is easy to see the kernel is an ideal. In fact, if α and β are two spark classes, and $\alpha \in \ker(\Pi_p)$, then we can choose representatives $(a, 0)$ and (b, s) for α and β respectively, with spark equations $da = e - 0$ and $db = f - s$, where a, e, f are smooth. By the product formula, $\alpha\beta$ can be represented by $(a \wedge f + (-1)^{\deg a + 1} 0 \wedge b, 0 \wedge s) = (a \wedge f, 0)$ which is in $\ker(\Pi_p)$.

Hence, $\hat{\mathbf{H}}^*(X, p)$ carries a ring structure induced from $\hat{\mathbf{H}}^*(X)$.

$\square$

3.2. Functoriality.

Proposition 3.8. *There are commutative diagrams*

$$\begin{array}{ccc} \hat{\mathbf{H}}^k(X) & \xrightarrow{\delta_1} & \mathcal{Z}_{\mathbb{Z}}^{k+1}(X) \\ \downarrow{\scriptstyle \Pi_p} & & \downarrow{\scriptstyle \pi_p} \\ \hat{\mathbf{H}}^k(X, p) & \xrightarrow{\delta_1} & \mathcal{Z}_{\mathbb{Z}}^{k+1}(X, p) \end{array} \qquad\qquad \begin{array}{ccc} \hat{\mathbf{H}}^k(X) & \xrightarrow{\delta_2} & H^{k+1}(X, \mathbb{Z}) \\ \downarrow{\scriptstyle \Pi_p} & & \parallel{\scriptstyle =} \\ \hat{\mathbf{H}}^k(X, p) & \xrightarrow{\delta_2} & H^{k+1}(X, \mathbb{Z}) \end{array}$$

Proof. Let $\alpha \in \hat{\mathbf{H}}^k(X)$. Choose a representative $(a, r) \in \alpha$ with spark equation $da = e - r$. Then $\pi_p \circ \delta_1(\alpha) = \pi_p(e)$, and $\delta_1 \circ \Pi_p(\alpha) = \delta_1 \circ \Pi_p([(a, r)]) = \delta_1([(\pi_p(a), r)]) = \pi_p(e)$ since $(\pi_p(a), r)$ is a $\bar{d}$-spark of level p with spark equation $d_p(\pi_p a) = \pi_p(e) - \Psi_p r$. Hence, the first diagram is commutative. We can verify the second one by the same way.

$\square$

Moreover, we have the following theorem

Theorem 3.9. *Any holomorphic map* $f : X \to Y$ *between complex manifolds induces a graded ring homomorphism*

$$f^* : \hat{\mathbf{H}}^*(Y, p) \to \hat{\mathbf{H}}^*(X, p)$$

with the property that if $g : Y \to Z$ *is holomorphic, then* $(g \circ f)^* = f^* \circ g^*$.

Proof. It was shown in [HLZ] [HL2] that f induces a ring homomorphism $f^* : \hat{\mathbf{H}}^*(Y) \to \hat{\mathbf{H}}^*(X)$ with the asserted property. It suffices to show $f^*(\ker \Pi_p) \subset (\ker \Pi_p)$ which is directly from Lemma 3.7. $\square$

Corollary 3.10. $\hat{\mathbf{H}}^*(\bullet, p)$ *is a graded ring functor on the category of complex manifolds and holomorphic maps.*

Theorem 3.11. *(Gysin map) Any holomorphic map* $f : X^{m+r} \to Y^m$ *between complex manifolds induces a graded ring homomorphism*

$$f_* : \hat{\mathbf{H}}^*(X, p) \to \hat{\mathbf{H}}^{*-2r}(Y, p - r).$$

Proof. It was shown in [HLZ] [HL2] that f induces a Gysin map $f_* : \hat{\mathbf{H}}^*(X) \to \hat{\mathbf{H}}^{*-r}(Y)$. Moreover, from Lemma 3.7, it is plain to get $f^*(\ker \Pi_p) \subset (\ker \Pi_{p-r})$. $\square$

4. Čech-Dolbeault Sparks of Level p

We now consider other presentations of the $\bar{d}$-spark classes. We shall introduce the Čech-Dolbeault spark complex of level p which is a generalization of the Čech-Dolbeault spark complex in [HL3].

Recall that, for a complex manifold X, we can decompose the space of smooth k-forms over an open set $U \subset X$ by types:

$$\mathcal{E}^k(U) \equiv \bigoplus_{r+s=k} \mathcal{E}^{r,s}(U).$$

Let $\mathcal{E}^k(U, p) = \bigoplus_{r+s=k, r<p} \mathcal{E}^{r,s}(U)$, and $\mathcal{E}_p^k$ denote the subsheaf of $\mathcal{E}^k$ with $\mathcal{E}_p^k(U) = \mathcal{E}^k(U, p)$. And similarly, we can define the sheaf $\mathcal{D}_p'^k$ with $\mathcal{D}_p'^k(U) = \mathcal{D}'^k(U, p)$.

Suppose $\mathcal{U}$ is a good cover of X and consider the total complex of the following double complex with total differential $D_p = \delta + (-1)^r d_p$:

It is easy to see the row complexes are exact everywhere except in the first column on the left, and

$$\{\ker(\delta) \text{ on the left column}\} \cong \{\text{global sections of sheaves } \mathcal{E}_p^*\} = \mathcal{E}^*(X, p).$$

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Hence,

$$H^*(\bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{E}_p^s)) \cong H^*(\mathcal{E}^*(X, p)) \cong H^*(X, p).$$

Note that every column complex is exact everywhere except at the bottom and the level of p from the bottom.

Now we consider the triple of complexes

$$(F_p^*, E_p^*, I_p^*) \equiv (\bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{E}_p^s), \mathcal{E}^*(X, p), C^*(\mathcal{U}, \mathbb{Z})).$$

And we have

Proposition 4.1. *The triple (F_p^*, E_p^*, I_p^*) defined above is a spark complex (even in the sense of [HL2]), which is called the **Čech-Dolbeault spark complex of level p**, or the **smooth hyperspark complex of level p**.*

Proof. We have shown that $E_p^* \hookrightarrow F_p^*$ induces an isomorphism $H^*(E_p^*) \cong H^*(F_p^*)$. Also there is an injective cochain map $I_p^* \equiv C^*(\mathcal{U}, \mathbb{Z}) \hookrightarrow C^*(\mathcal{U}, \mathcal{E}_p^0) \hookrightarrow \bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{E}_p^s) \equiv F_p^*$.

$E_p^k \cap I_p^k = \{0\}$ for $k > 0$ is trivial. $\square$

Definition 4.2. *A **Čech-Dolbeault spark of level p** of degree k, or a **smooth hyperspark of level p** is an element*

$$a \in \bigoplus_{r+s=k} C^r(\mathcal{U}, \mathcal{E}_p^s)$$

with the spark equation

$$D_p a = e - r$$

where $e \in \mathcal{E}_p^{k+1}(X) \subset C^0(\mathcal{U}, \mathcal{E}_p^{k+1})$ is of bidegree $(0, k+1)$ and $r \in C^{k+1}(\mathcal{U}, \mathbb{Z})$.

*Two Čech-Dolbeault sparks of level p, a and a' are **equivalent** if there exist $b \in \bigoplus_{r+s=k-1} C^r(\mathcal{U}, \mathcal{E}_p^s)$ and $s \in C^k(\mathcal{U}, \mathbb{Z})$ satisfying*

$$a - a' = D_p b + s.$$

The equivalence class determined by a Čech-Dolbeault spark a will be denoted by $[a]$, and the group of Čech-Dolbeault spark classes of level p will be denoted by $\hat{\mathbf{H}}_{smooth}^k(X, p)$.

Recall that the **smooth hyperspark complex** ([HL2] [H1]) is defined by

$$(F^*, E^*, I^*) = (\bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{E}^s), \mathcal{E}^*(X), C^*(\mathcal{U}, \mathbb{Z})).$$

The associated group of smooth hyperspark classes is denoted by $\hat{\mathbf{H}}_{smooth}^*(X)$, whose ring structure was established in [H1]. The relation between the smooth hyperspark complex and the Čech-Dolbeault spark complex of level p is the same as the relation between the de Rham-Federer spark complex and the Dolbeault-Federer spark complex of level p. We have the natural morphism $(\pi_p, \pi_p, id) : (F^*, E^*, I^*) \longrightarrow (F_p^*, E_p^*, I_p^*)$. Explicitly, we have the following commutative diagram

$$
\begin{array}{ccccc}
C^*(\mathcal{U}, \mathbb{Z}) & \xrightarrow{\ i\ } & \bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{E}^s) & \xleftarrow{\ i\ } & \mathcal{E}^*(X) \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \pi_p} & & \downarrow{\scriptstyle \pi_p} \\
C^*(\mathcal{U}, \mathbb{Z}) & \xrightarrow{\ i\ } & \bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{E}_p^s) & \xleftarrow{\ i\ } & \mathcal{E}^*(X, p)
\end{array}
$$

Theorem 4.3. *The morphism of spark complexes $(\pi_p, \pi_p, id) : (F^*, E^*, I^*) \to (F_p^*, E_p^*, I_p^*)$ induces a surjective group homomorphism*

$$\Pi_p : \hat{\mathbf{H}}_{smooth}^*(X) \to \hat{\mathbf{H}}_{smooth}^*(X, p)$$

whose kernel is an ideal. Hence, $\hat{\mathbf{H}}^*_{smooth}(X, p)$ carries a ring structure.

Proof. The proof is similar to Theorem 3.6. It's plain to see that the diagram above commutes and π_p commutes with differentials. Hence, the induced map $a \mapsto \pi_p(a)$ on sparks descends to a group homomorphism $\Pi_p : \hat{\mathbf{H}}^k_{smooth}(X) \to \hat{\mathbf{H}}^k_{smooth}(X, p)$.

To prove the surjectivity, consider a spark $a \in F_p^k$ with $D_p a = e - r$ for some $e \in E_p^{k+1}$ and $r \in I_p^{k+1} = I^{k+1}$. We can choose some smooth form which represents same cohomology class with r in $H^{k+1}(F^*) \cong H^{k+1}(E^*)$, so there exist $a_0 \in F^k$, $e_0 \in E^{k+1}$ such that $Da_0 = e_0 - r$. We have

$$\pi_p(Da_0) = \pi_p(e_0) - r \Rightarrow D_p(\pi_p a_0) = \pi_p e_0 - r.$$

Hence, $D_p(a - \pi_p a_0) = e - \pi_p e_0$ is a smooth form. It follows by [HL2, Lemma 1.5] that there exist $b \in F_p^{k-1}$ and $f \in E_p^k$ with $a - \pi_p a_0 = f + D_p b$. Set $\tilde{a} = a_0 + f + Db$ and note that $D\tilde{a} = Da_0 + Df + DDb = e_0 - r + df = (e_0 + df) - r$. Hence, $\tilde{a}$ is a spark of degree k and $\pi_p \tilde{a} = \pi_p(a_0 + f + Db) = \pi_p a_0 + f + D_p b = a$. So Π_p is surjective.

We need the following lemma to show the kernel is an ideal.

Lemma 4.4. *On $\hat{\mathbf{H}}^k_{smooth}(X)$, one has that $\ker(\Pi_p) = \{\alpha \in \hat{\mathbf{H}}^k_{smooth}(X) : \exists a \in \alpha$ where $a \in \mathcal{E}^k(X) \subset C^0(\mathcal{U}, \mathcal{E}^k)$ and $\pi_p(a) = 0\}$. In particular, $\ker \Pi_p \subset \hat{\mathbf{H}}^k_\infty(X)$.*

Proof. One direction is clear. Suppose $\alpha \in \ker(\Pi_p)$ and choose any spark $a \in \alpha$ with $Da = e - r$. $\Pi_p(\alpha) = 0$ means that there exist $b \in F_p^{k-1}$ and $s \in I_p^k = I^k$ with $\pi_p(a) = D_p b + s = \pi_p(Db) + s$ which implies $D_p(\pi_p a) = \delta s$. On the other hand, $D_p(\pi_p a) = \pi_p(Da) = \pi_p e - r$. So we have $\pi_p e = 0$ and $-r = \delta s$. Replace a by $\bar{a} = a - Db - s$, then $\bar{a}$ represents the same class as a and $\pi_p(\bar{a}) = \pi_p(a - Db) - s = 0$.

In fact, we can choose $\bar{a}$ in $\mathcal{E}^k(X) \subset C^0(\mathcal{U}, \mathcal{E}^k)$. Since

$$D\bar{a} = Da - \delta s = e - (r + \delta s) = e$$

is a global smooth form, it follows by [HL2, Lemma 1.5] and the fact

$$H^*\left(\bigoplus_{r+s=*} C^r(\mathcal{U}, F^p \mathcal{E}^s)\right) \cong H^*(F^p \mathcal{E}^*(X))$$

that we can choose $\bar{a}$ to be smooth. Note that $F^0 \mathcal{E}^* \supset F^1 \mathcal{E}^* \supset \cdots \supset F^p \mathcal{E}^* \supset \cdots$ is the naive filtration. $\square$

By the product formula of $\hat{\mathbf{H}}^*_{smooth}(X)$ [H1], it is easy to see the kernel is an ideal. Hence, $\hat{\mathbf{H}}^*_{smooth}(X, p)$ carries a ring structure induced from $\hat{\mathbf{H}}^*_{smooth}(X)$.

$\square$

5. Čech-Dolbeault Hypersparks of Level p

Now we introduce the Čech-Dolbeault hyperspark complex of level p which set up a bridge connecting the Čech-Dolbeault spark complex of level p and the Dolbeault-Federer spark complex of level p.

Fix a good cover $\mathcal{U}$ of X and consider total complex of the following double complex with total differential $D_p = \delta + (-1)^r d_p$:

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & & \vdots & \\
\uparrow{\scriptstyle d_p} & & \uparrow{\scriptstyle -d_p} & & \uparrow{\scriptstyle d_p} & & & \uparrow{\scriptstyle (-1)^r d_p} & \\
C^0(\mathcal{U},\mathcal{D}_p'^2) \xrightarrow{\delta} & C^1(\mathcal{U},\mathcal{D}_p'^2) \xrightarrow{\delta} & C^2(\mathcal{U},\mathcal{D}_p'^2) \xrightarrow{\delta} \cdots & \cdots \xrightarrow{\delta} & C^r(\mathcal{U},\mathcal{D}_p'^2) \xrightarrow{\delta} \cdots \\
\uparrow{\scriptstyle d_p} & & \uparrow{\scriptstyle -d_p} & & \uparrow{\scriptstyle d_p} & & & \uparrow{\scriptstyle (-1)^r d_p} & \\
C^0(\mathcal{U},\mathcal{D}_p'^1) \xrightarrow{\delta} & C^1(\mathcal{U},\mathcal{D}_p'^1) \xrightarrow{\delta} & C^2(\mathcal{U},\mathcal{D}_p'^1) \xrightarrow{\delta} \cdots & \cdots \xrightarrow{\delta} & C^r(\mathcal{U},\mathcal{D}_p'^1) \xrightarrow{\delta} \cdots \\
\uparrow{\scriptstyle d_p} & & \uparrow{\scriptstyle -d_p} & & \uparrow{\scriptstyle d_p} & & & \uparrow{\scriptstyle (-1)^r d_p} & \\
C^0(\mathcal{U},\mathcal{D}_p'^0) \xrightarrow{\delta} & C^1(\mathcal{U},\mathcal{D}_p'^0) \xrightarrow{\delta} & C^2(\mathcal{U},\mathcal{D}_p'^0) \xrightarrow{\delta} \cdots & \cdots \xrightarrow{\delta} & C^r(\mathcal{U},\mathcal{D}_p'^0) \xrightarrow{\delta} \cdots
\end{array}
$$

It is easy to see the row complexes are exact everywhere except the first column on the left, and

$$\{\ker(\delta) \text{ on the left column}\} \cong \{\text{global sections of sheaves } \mathcal{D}_p'^*\} = \mathcal{D}'^*(X,p).$$

Hence,

$$H^*\left(\bigoplus_{r+s=*} C^r(\mathcal{U},\mathcal{D}_p'^s)\right) \cong H^*(\mathcal{D}'^*(X,p)) \cong H^*(\mathcal{E}^*(X,p)) \cong H^*(X,p).$$

Note that every column complex is exact everywhere except at the bottom and the level of p from the bottom.

Now we consider the triple of complexes

$$(F_p^*, E_p^*, I_p^*) \equiv \left(\bigoplus_{r+s=*} C^r(\mathcal{U},\mathcal{D}_p'^s), \mathcal{E}^*(X,p), \bigoplus_{r+s=*} C^r(\mathcal{U},\mathcal{IF}^s)\right).$$

And we have

Proposition 5.1. *The triple of complexes (F_p^*, E_p^*, I_p^*) as defined above is a spark complex, which is called the **Čech-Dolbeault hyperspark complex of level p**, or more simply, the **hyperspark complex of level p**.*

Proof. We have shown that $E_p^* \hookrightarrow F_p^*$ induces an isomorphism $H^*(E_p^*) \cong H^*(F_p^*)$. Also there is a map

$$\Psi_p : I_p^* \equiv \bigoplus_{r+s=*} C^r(\mathcal{U},\mathcal{IF}^s) \hookrightarrow \bigoplus_{r+s=*} C^r(\mathcal{U},\mathcal{D}'^s) \xrightarrow{\pi_p} \bigoplus_{r+s=*} C^r(\mathcal{U},\mathcal{D}_p'^s) \equiv F_p^*.$$

And $E_p^k \cap I_p^k = \{0\}$ for $k > 0$ follows [HL3, Appendix B]. $\square$

Definition 5.2. *A **Čech-Dolbeault hyperspark of level p** of degree k, or **hyperspark of level p** is a pair*

$$(a,r) \in \bigoplus_{r+s=k} C^r(\mathcal{U},\mathcal{D}_p'^s) \oplus \bigoplus_{r+s=k+1} C^r(\mathcal{U},\mathcal{IF}^s)$$

with the spark equations

$$D_p a = e - \Psi_p r \quad and \quad Dr = 0$$

where $e \in \mathcal{E}_p^{k+1}(X) \subset C^0(\mathcal{U},\mathcal{D}_p'^{k+1})$ is of bidegree $(0, k+1)$.

*Two Čech-Dolbeault sparks of level p, (a,r) and (a',r') are **equivalent** if there exist*

$$b \in \bigoplus_{r+s=k-1} C^r(\mathcal{U},\mathcal{D}_p'^s) \quad and \quad s \in \bigoplus_{r+s=k} C^r(\mathcal{U},\mathcal{IF}^s)$$

satisfying

$$a - a' = D_p b + s \quad and \quad r = -Ds.$$

The equivalence class determined by a Čech-Dolbeault hyperspark (a,r) will be denoted by $[(a,r)]$, and the group of Čech-Dolbeault hyperspark classes of level p will be denoted by $\hat{\mathbf{H}}^k_{hyperspark}(X,p)$.

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Harvey and Lawson introduced the **hyperspark complex**

$$(F^*, E^*, I^*) = (\bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{D}'^s), \mathcal{E}^*(X), \bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{I}\mathcal{F}^s))$$

in [HL2]. The hyperspark complex and the Čech-Dolbeault hyperspark complex of level p is related by the natural morphism $(\pi_p, \pi_p, id) : (F^*, E^*, I^*) \longrightarrow (F_p^*, E_p^*, I_p^*)$. Explicitly, we have the following commutative diagram

$$
\begin{array}{ccccc}
\bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{I}\mathcal{F}^s) & \xrightarrow{\ i\ } & \bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{D}'^s) & \xleftarrow{\ i\ } & \mathcal{E}^*(X) \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \pi_p} & & \downarrow{\scriptstyle \pi_p} \\
\bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{I}\mathcal{F}^s) & \xrightarrow{\ i\ } & \bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{D}'^s_p) & \xleftarrow{\ i\ } & \mathcal{E}^*(X, p)
\end{array}
$$

Similar to last two sections, we have the following lemma and theorem

Lemma 5.3. *On* $\hat{\mathbf{H}}^k_{hyperspark}(X)$, *one has that* $\ker(\Pi_p) = \{\alpha \in \hat{\mathbf{H}}^k_{hyperspark}(X) : \exists a \in \alpha \text{ where } a \in \mathcal{E}^k(X) \subset C^0(\mathcal{U}, \mathcal{D}'^k) \text{ and } \pi_p(a) = 0\}$. *In particular,* $\ker(\Pi_p) \subset \hat{\mathbf{H}}^k_\infty(X)$.

Theorem 5.4. *The morphism of spark complexes* $(\pi_p, \pi_p, id) : (F^*, E^*, I^*) \to (F_p^*, E_p^*, I_p^*)$ *induces a surjective group homomorphism*

$$\Pi_p : \hat{\mathbf{H}}^*_{hyperspark}(X) \to \hat{\mathbf{H}}^*_{hyperspark}(X, p)$$

whose kernel is an ideal. Hence, $\hat{\mathbf{H}}^*_{hyperspark}(X, p)$ *carries a ring structure.*

Harvey and Lawson showed

Theorem 5.5. [HL2] *Both the de Rham-Federer spark complex and the smooth hyperspark complex are quasi-isomorphic to the hyperspark complex. Hence,*

$$\hat{\mathbf{H}}^*_{spark}(X) \cong \hat{\mathbf{H}}^*_{hyperspark}(X) \cong \hat{\mathbf{H}}^*_{smooth}(X).$$

Similarly, we establish relations among the Dolbeault-Federer spark complex, the Čech-Dolbeault spark complex and the Čech-Dolbeault hyperspark complex of level p.

Theorem 5.6. *We have morphisms of spark complexes*

$$
\begin{array}{ccccc}
\left\{\begin{array}{c}\text{the de Rham-Federer}\\\text{spark complex}\end{array}\right\} & \xrightarrow{\ i\ } & \{\text{the hyperspark complex}\} & \xleftarrow{\ i\ } & \{\text{the smooth hyperspark complex}\} \\
\downarrow{\scriptstyle \pi_p} & & \downarrow{\scriptstyle \pi_p} & & \downarrow{\scriptstyle \pi_p} \\
\left\{\begin{array}{c}\text{the Dolbeault-Federer}\\\text{spark complex of level } p\end{array}\right\} & \xrightarrow{\ i\ } & \left\{\begin{array}{c}\text{the Čech-Dolbeault hyperspark}\\\text{complex of level } p\end{array}\right\} & \xleftarrow{\ i\ } & \left\{\begin{array}{c}\text{the Čech-Dolbeault spark}\\\text{complex of level } p\end{array}\right\}
\end{array}
$$

where horizontal morphisms are quasi-isomorphisms.

Hence we get induced homomorphisms

$$
\begin{array}{ccccc}
\hat{\mathbf{H}}^*_{spark}(X) & \xrightarrow{\ =\ } & \hat{\mathbf{H}}^*_{hyperspark}(X) & \xleftarrow{\ =\ } & \hat{\mathbf{H}}^*_{smooth}(X) \\
\downarrow{\scriptstyle \Pi_p} & & \downarrow{\scriptstyle \Pi_p} & & \downarrow{\scriptstyle \Pi_p} \\
\hat{\mathbf{H}}^*_{spark}(X, p) & \xrightarrow{\ =\ } & \hat{\mathbf{H}}^*_{hyperspark}(X, p) & \xleftarrow{\ =\ } & \hat{\mathbf{H}}^*_{smooth}(X, p)
\end{array}
$$

where the horizontal ones are isomorphism.

Proof. It is easy to see we have the following two commutative diagrams

$$
\begin{array}{ccccc}
\mathcal{I}\mathcal{F}^*(X) & \xrightarrow{\ \Psi_p\ } & \mathcal{D}'^*(X, p) & \xleftarrow{\ i\ } & \mathcal{E}^*(X, p) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \| \\
\bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{I}\mathcal{F}^s) & \xrightarrow{\ \Psi_p\ } & \bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{D}'^s_p) & \xleftarrow{\ i\ } & \mathcal{E}^*(X, p)
\end{array}
$$

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and

$$
\begin{array}{ccc}
C^*(\mathcal{U},\mathbb{Z}) \xrightarrow{\;\;i\;\;} \bigoplus_{r+s=*} C^r(\mathcal{U},\mathcal{E}_p^s) \xleftarrow{\;\;i\;\;} \mathcal{E}^*(X,p) \\
\Big\downarrow{\scriptstyle i} \qquad\qquad \Big\downarrow{\scriptstyle i} \qquad\qquad \Big\| \\
\bigoplus_{r+s=*} C^r(\mathcal{U},\mathcal{IF}^s) \xrightarrow{\;\Psi_p\;} \bigoplus_{r+s=*} C^r(\mathcal{U},\mathcal{D}_p'^s) \xleftarrow{\;\;i\;\;} \mathcal{E}^*(X,p)
\end{array}
$$

where

$$
i : \mathcal{IF}^*(X) \longrightarrow \bigoplus_{r+s=*} C^r(\mathcal{U},\mathcal{IF}^s) \quad \text{and} \quad i : C^*(\mathcal{U},\mathbb{Z}) \longrightarrow \bigoplus_{r+s=*} C^r(\mathcal{U},\mathcal{IF}^s)
$$

are quasi-isomorphisms of cochain complexes.

$\square$

So far, we have introduced three spark complexes associated to a complex manifold X, and showed the natural isomorphisms between the groups of spark classes associated to them. We denote the groups of spark classes by $\hat{\mathbf{H}}^*(X,p)$ collectively, and call them the **Harvey-Lawson spark characters of level p associated to** X. The ring structure of $\hat{\mathbf{H}}^*(X,p)$ is induced from the ring structure of $\hat{\mathbf{H}}^*(X)$. There are two different ways to define the product in $\hat{\mathbf{H}}^*(X)$. Harvey, Lawson and Zweck [HLZ] defined the product via the de Rham-Federer spark complex. And the author defined the product via the smooth hyperspark complex in [H1] and showed two ring structures are equivalent. In the next section, we shall define the product for Deligne cohomology by both theories.

6. The Ring Structure on Deligne Cohomology with Analytic Formula

Deligne cohomology was invented by Deligne in 1970's. In [B], Beilinson defined the ring structure on Deligne cohomology. In 1995, Deninger [D] defined higher operations — Massey products in Deligne cohomology. In this section, we shall give a product formula for Deligne cohomology via spark theory. A construction of Massey products in Deligne cohomology will appear in [H2].

We follow [EV] to define Deligne cohomology and its ring structure.

Definition 6.1. *Let X be a complex manifold. For $p \geq 0$, the Deligne complex $\mathbb{Z}_{\mathcal{D}}(p)$ is the complex of sheaves:*

$$
0 \to \mathbb{Z} \xrightarrow{\;i\;} \Omega^0 \xrightarrow{\;d\;} \Omega^1 \xrightarrow{\;d\;} \cdots \xrightarrow{\;d\;} \Omega^{p-1} \to 0
$$

where Ω^k denotes the sheaf of holomorphic k-forms on X. The hypercohomology groups $\mathbb{H}^q(X,\mathbb{Z}_{\mathcal{D}}(p))$ are called the Deligne cohomology groups of X, and are denoted by $H_{\mathcal{D}}^q(X,\mathbb{Z}(p))$.

Remark 6.2. *In Deligne complex $\mathbb{Z}_{\mathcal{D}}(p)$, we always consider that $\mathbb{Z}$ is of degree 0, and Ω^k is of degree $k+1$.*

Example 6.3. *It is easy to see $H_{\mathcal{D}}^q(X,\mathbb{Z}(0)) = H^q(X,\mathbb{Z})$ and $H_{\mathcal{D}}^q(X,\mathbb{Z}(1)) = H^{q-1}(X,\mathcal{O}^*)$.*

In [B], Beilinson defined a cup product

$$
\cup : \mathbb{Z}_{\mathcal{D}}(p) \otimes \mathbb{Z}_{\mathcal{D}}(p') \to \mathbb{Z}_{\mathcal{D}}(p+p')
$$

by

$$
x \cup y = \begin{cases} x \cdot y & \text{if } \deg x = 0; \\ x \wedge dy & \text{if } \deg x > 0 \text{ and } \deg y = p'; \\ 0 & \text{otherwise.} \end{cases}
$$

The cup product $\cup$ is a morphism of complexes and associative [EV] [Br1], hence induces a ring structure on

$$
\bigoplus_{p,q} H_{\mathcal{D}}^q(X,\mathbb{Z}(p)).
$$

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We are identifying the Deligne cohomology groups with subgroups of the groups of $\bar{d}$-spark classes. Then we give a product formula for Deligne cohomology.

Lemma 6.4. *We have the short exact sequence*

$$0 \to H_{\mathcal{D}}^k(X, \mathbb{Z}(p)) \to \hat{\mathbf{H}}_{spark}^{k-1}(X, p) \to \mathcal{Z}_{\mathbb{Z}}^k(X, p) \to 0$$

which is the middle row of 3×3 diagram for the group of $\bar{d}$-spark classes of level p. Hence, for any Deligne class $\alpha \in H_{\mathcal{D}}^k(X, \mathbb{Z}(p))$, there exists a spark representative

$$(a, r) \in \mathcal{D}'^{k-1}(X, p) \oplus \mathcal{IF}^k(X) \quad with \quad d_p a = -\Psi_p(r), \quad dr = 0.$$

Proof. By Proposition 2.8, it suffices to show

$$H_{\mathcal{D}}^k(X, \mathbb{Z}(p)) \cong H^{k-1}(\mathrm{Cone}(\Psi : \mathcal{IF}^*(X) \to \mathcal{D}'^*(X, p))).$$

By definition, $H_{\mathcal{D}}^*(X, \mathbb{Z}(p))$ is the hypercohomology of the complex of sheaves

$$0 \to \mathbb{Z} \to \Omega^0 \to \Omega^1 \to \dots \to \Omega^{p-1} \to 0.$$

In other words, it is the hypercohomology of the Cone $(\mathbb{Z} \to \Omega^{*<p})[-1]$.

Consider the acyclic resolutions:

$$\mathbb{Z} \to \mathcal{IF}^* \quad and \quad \Omega^k \to \mathcal{D}'^{k,*}.$$

And we have quasi-isomorphism of complexes of sheaves

$$\mathrm{Cone}(\mathbb{Z} \to \Omega^{*<p}) \simeq \mathrm{Cone}(\Psi : \mathcal{IF}^* \to \bigoplus_{s+t=*, s<p} \mathcal{D}'^{s,t}),$$

and hence

$$H_{\mathcal{D}}^k(X, \mathbb{Z}(p)) \cong \mathbb{H}^{k-1}(\mathrm{Cone}(\mathbb{Z} \to \Omega^{*<p})) \cong \mathbb{H}^{k-1}(\mathrm{Cone}(\Psi : \mathcal{IF}^* \to \bigoplus_{s+t=*, s<p} \mathcal{D}'^{s,t}))$$

$$\cong H^{k-1}(\mathrm{Cone}(\Psi : \mathcal{IF}^*(X) \to \mathcal{D}'^*(X, p))).$$

Then for any Deligne class $\alpha \in H_{\mathcal{D}}^k(X, \mathbb{Z}(p)) \subset \hat{\mathbf{H}}^{k-1}(X, p)$, we can find a representative $(a, r) \in \mathcal{D}'^{k-1}(X, p) \oplus \mathcal{I}^k(X)$ with $d_p a = e - \Psi_p(r)$, $dr = 0$. And we have $e = 0$ since $\alpha \in \ker \delta_1$.
□

Applying the representation of Deligne cohomology classes in terms of currents above, we define a product in Deligne cohomology

$$H_{\mathcal{D}}^k(X, \mathbb{Z}(p)) \otimes H_{\mathcal{D}}^l(X, \mathbb{Z}(q)) \longrightarrow H_{\mathcal{D}}^{k+l}(X, \mathbb{Z}(p+q)).$$

First, for any Deligne class $\alpha \in H_{\mathcal{D}}^k(X, \mathbb{Z}(p))$, we choose a spark representative

$$(a, r) \in \mathcal{D}'^{k-1}(X, p) \oplus \mathcal{IF}^k(X) \quad with \quad d_p a = -\Psi_p(r), \quad dr = 0.$$

Similarly, for any $\beta \in H_{\mathcal{D}}^l(X, \mathbb{Z}(q))$, we choose a spark representative

$$(b, s) \in \mathcal{D}'^{l-1}(X, q) \oplus \mathcal{IF}^l(X) \quad with \quad d_q b = -\Psi_q(s), \quad ds = 0.$$

Since $\Pi_p : \hat{\mathbf{H}}^*(X) \to \hat{\mathbf{H}}^*(X, p)$ is surjective, there exist

$$(\tilde{a}, r) \in \mathcal{D}'^{k-1}(X) \oplus \mathcal{IF}^k(X) \quad with \quad \Pi_p[(\tilde{a}, r)] = [(\pi_p(\tilde{a}), r)] = [(a, r)] = \alpha,$$

and

$$(\tilde{b}, s) \in \mathcal{D}'^{l-1}(X) \oplus \mathcal{IF}^l(X) \quad with \quad \Pi_q[(\tilde{b}, s)] = [(\pi_q(\tilde{b}), s)] = [(b, s)] = \beta.$$

Write the spark equations for $\tilde{a}$ and $\tilde{b}$ as

$$d\tilde{a} = e - r \quad and \quad d\tilde{b} = f - s,$$

where $\pi_p \tilde{a} = a$, $\pi_p e = 0$ and $\pi_q \tilde{b} = b$, $\pi_q f = 0$.

By the product formula [HL2, Theorem 3.5]

$$[(\tilde{a}, r)] * [(\tilde{b}, s)] = [(\tilde{a} \wedge f + (-1)^k r \wedge \tilde{b}, r \wedge s)] = [(\tilde{a} \wedge s + (-1)^k e \wedge \tilde{b}, r \wedge s)].$$

Since

$$d(\tilde{a} \wedge f + (-1)^k r \wedge \tilde{b}) = e \wedge f - r \wedge s \quad \text{and} \quad \pi_{p+q}(e \wedge f) = 0,$$

we get $\Pi_{p+q}[(\tilde{a} \wedge f + (-1)^k r \wedge \tilde{b}, r \wedge s)] \in H_{\mathcal{D}}^{k+l}(X, \mathbb{Z}(p+q))$, and define it as $\alpha * \beta$.

Now we show the product is well-defined, i.e. it is independent of the choices of representatives α and β. If we have another representative $(a', r') \in \alpha$ and a lift $(\tilde{a}', r')$ with $\Pi_p[(\tilde{a}', r')] = [(a', r')] = \alpha$, then $[(\tilde{a}, r) - (\tilde{a}', r')] \in \ker \Pi_p$. By Lemma 3.7, there exists a representative of spark class $[(\tilde{a}, r) - (\tilde{a}', r')]$, which is of form $(c, 0)$ where c is smooth and $\pi_p(c) = 0$. Then we have

$$
\begin{aligned}
&\Pi_{p+q}([(\tilde{a}, r)] * [(\tilde{b}, s)] - [(\tilde{a}', r')] * [(\tilde{b}, s)]) \\
=\ & \Pi_{p+q}([(\tilde{a}, r) - (\tilde{a}', r')] * [(\tilde{b}, s)]) \\
=\ & \Pi_{p+q}([(c, 0)] * [(\tilde{b}, s)]) \\
=\ & \Pi_{p+q}([(c \wedge f + (-1)^k 0 \wedge \tilde{b}, 0)]) \\
=\ & 0
\end{aligned}
$$

Similarly, we can show the product does not depend on representatives of β either.

Remark 6.5. *In the process above, we can always choose good representatives a, $\tilde{a}$, b, $\tilde{b}$, r and s in sense of [HL2, Proposition 3.1] such that all wedge products are well defined.*

Theorem 6.6. Product formula of Deligne cohomology I
For any Deligne class $\alpha \in H_{\mathcal{D}}^k(X, \mathbb{Z}(p))$ and $\beta \in H_{\mathcal{D}}^l(X, \mathbb{Z}(q))$, there exist spark representations (a, r) for α and (b, s) for β as above. Let $(\tilde{a}, r)$ and $(\tilde{b}, s)$ be de Rham-Federer sparks which are lifts of (a, r) and (b, s). Then

$$\alpha * \beta = \Pi_{p+q}[(\tilde{a} \wedge s + (-1)^k e \wedge \tilde{b}, r \wedge s)] = [(\pi_{p+q}(\tilde{a} \wedge s + (-1)^k e \wedge b), r \wedge s)] \in H_{\mathcal{D}}^{k+l}(X, \mathbb{Z}(p+q)).$$

Proof. We have shown the product is well defined. In Theorem 6.11, we shall verify that this product is equivalent to Beilinson's definition. $\qquad\square$

Remark 6.7. *Suppose X is a algebraic manifold and $CH^*(X)$ is the Chow ring of X. Considering every nonsingular subvariety as a integrally flat current, we can define the group homomorphism*

$$\psi : CH^p(X) \to H_{\mathcal{D}}^{2p}(X, \mathbb{Z}(p)).$$

By our product formula, it is quite easy to see this map induces a ring homomorphism, i.e. the ring structure of Deligne cohomology is compatible with the ring structure of the Chow ring. We shall explain this in the next section.

Now we rewrite last theorem in terms of the Čech-Dolbeault spark complex and give a similar formula. Then we show this product is equivalent to the product defined by Beilinson in Theorem 6.11.

Lemma 6.8. *We have the short exact sequence*

$$0 \to H_{\mathcal{D}}^k(X, \mathbb{Z}(p)) \to \hat{\mathbf{H}}_{smooth}^{k-1}(X, p) \to \mathcal{Z}_{\mathbb{Z}}^k(X, p) \to 0.$$

For any Deligne class $\alpha \in H_{\mathcal{D}}^k(X, \mathbb{Z}(p))$, there exists a representative

$$(a, r) \in \bigoplus_{r+s=k-1} C^r(\mathcal{U}, \mathcal{E}_p^s) \oplus C^k(\mathcal{U}, \mathbb{Z}) \quad \text{with} \quad D_p a = -r, \quad \delta r = 0.$$

Proof. Note that we use (a, r) to represent a Čech-Dolbeault spark here although we can omit r. The reason is that we can make the proof of Theorem 6.11 clearer with this representation.

Applying the following quasi-isomorphisms of complexes of sheaves:

$$\mathbb{Z} \simeq \mathcal{C}^*(\mathcal{U}, \mathbb{Z}) \quad \text{and} \quad \Omega^{*<p} \simeq \mathcal{E}_p^* \simeq \bigoplus_{r+s=*} \mathcal{C}^r(\mathcal{U}, \mathcal{E}_p^s),$$

we get

$$H_{\mathcal{D}}^k(X, \mathbb{Z}(p)) \cong \mathbb{H}^{k-1}(\text{Cone}(\mathbb{Z} \to \Omega^{*<p}) \cong \mathbb{H}^{k-1}(\text{Cone}(\mathcal{C}^*(\mathcal{U}, \mathbb{Z}) \to \bigoplus_{r+s=*} \mathcal{C}^r(\mathcal{U}, \mathcal{E}_p^s)))$$

$$\cong H^{k-1}(\text{Cone}(C^*(\mathcal{U}, \mathbb{Z}) \to \bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{E}_p^s))).$$

By Proposition 2.8 and definition of $\hat{\mathbf{H}}_{smooth}^*(X, p)$, we have the short exact sequence:

$$0 \to H_{\mathcal{D}}^k(X, \mathbb{Z}(p)) \to \hat{\mathbf{H}}_{smooth}^{k-1}(X, p) \to \mathcal{Z}_{\mathbb{Z}}^k(X, p) \to 0.$$

Hence, for any Deligne class $\alpha \in H_{\mathcal{D}}^k(X, \mathbb{Z}(p)) \subset \hat{\mathbf{H}}_{smooth}^{k-1}(X, p)$, we can find a representative $(a, r) \in \bigoplus_{r+s=k-1} C^r(\mathcal{U}, \mathcal{E}_p^s) \oplus C^k(\mathcal{U}, \mathbb{Z})$ with $D_p a = e - r$, $\delta r = 0$. And we have $e = 0$ since $\alpha \in \ker \delta_1$. $\qquad \square$

Via the Čech-Dolbeault spark complex, we establish another product formula for Deligne cohomology.

Our goal is to define the product in Deligne cohomology

$$H_{\mathcal{D}}^k(X, \mathbb{Z}(p)) \otimes H_{\mathcal{D}}^l(X, \mathbb{Z}(q)) \longrightarrow H_{\mathcal{D}}^{k+l}(X, \mathbb{Z}(p+q)).$$

First, we choose a Čech-Dolbeault spark representative

$$(a, r) \in \bigoplus_{r+s=k-1} C^r(\mathcal{U}, \mathcal{E}_p^s) \oplus C^k(\mathcal{U}, \mathbb{Z}) \quad \text{with} \quad D_p a = -r, \quad \delta r = 0$$

for Deligne class $\alpha \in H_{\mathcal{D}}^k(X, \mathbb{Z}(p))$, and a Čech-Dolbeault spark representative

$$(b, s) \in \bigoplus_{r+s=l-1} C^r(\mathcal{U}, \mathcal{E}_p^s) \oplus C^l(\mathcal{U}, \mathbb{Z}) \quad \text{with} \quad D_q b = -s, \quad \delta s = 0$$

for $\beta \in H_{\mathcal{D}}^l(X, \mathbb{Z}(q))$.

Since $\Pi_p : \hat{\mathbf{H}}_{smooth}^*(X) \to \hat{\mathbf{H}}_{smooth}^*(X, p)$ is surjective, there exist smooth hypersparks

$$(\tilde{a}, r) \in \bigoplus_{r+s=k-1} C^r(\mathcal{U}, \mathcal{E}^s) \oplus C^k(\mathcal{U}, \mathbb{Z}) \quad \text{with} \quad \Pi_p[(\tilde{a}, r)] = [(a, r)] = \alpha,$$

and

$$(\tilde{b}, s) \in \bigoplus_{r+s=l-1} C^r(\mathcal{U}, \mathcal{E}^s) \oplus C^l(\mathcal{U}, \mathbb{Z}) \quad \text{with} \quad \Pi_q[(\tilde{b}, s)] = [(b, s)] = \beta.$$

Write the spark equations of $\tilde{a}$ and $\tilde{b}$ as

$$D\tilde{a} = e - r \quad \text{and} \quad D\tilde{b} = f - s,$$

where $\pi_p \tilde{a} = a$, $\pi_p e = 0$ and $\pi_q \tilde{b} = b$, $\pi_q f = 0$.

By the product formula of $\hat{\mathbf{H}}_{smooth}^*(X)$ constructed in [H1],

$$[(\tilde{a}, r)] * [(\tilde{b}, s)] = [\tilde{a} \cup f + (-1)^k r \cup \tilde{b}, r \cup s] = [\tilde{a} \cup s + (-1)^k e \cup \tilde{b}, r \cup s].$$

We have

$$D(\tilde{a} \cup f + (-1)^k r \cup \tilde{b}) = e \wedge f - r \cup s \text{ and } \pi_{p+q}(e \wedge f) = 0,$$

so we get $\Pi_{p+q}[(\tilde{a} \cup f + (-1)^k r \cup \tilde{b}, r \cup s)] \in H_{\mathcal{D}}^{k+l}(X, \mathbb{Z}(p+q))$, which is defined to be $\alpha * \beta$.

The product is only dependent on the spark classes α and β. If we have another representative $(a', r') \in \alpha$ and a lift $(\tilde{a}', r')$ with $\Pi_p[(\tilde{a}', r')] = [(a', r')] = \alpha$, then $[(\tilde{a}, r) - (\tilde{a}', r')] \in \ker \Pi_p$. By Lemma 4.4, we can choose a representative of spark class $[(\tilde{a}, r) - (\tilde{a}', r')]$, which is of form $(c, 0)$ where c is smooth and $\pi_p(c) = 0$. Then we have

$$
\begin{aligned}
& \Pi_{p+q}([(\tilde{a}, r)] * [(\tilde{b}, s)] - [(\tilde{a}', r')] * [(\tilde{b}, s)]) \\
= \ & \Pi_{p+q}([(\tilde{a}, r) - (\tilde{a}', r')] * [(\tilde{b}, s)]) \\
= \ & \Pi_{p+q}([(c, 0)] * [(\tilde{b}, s)]) \\
= \ & \Pi_{p+q}([(c \cup f + (-1)^k 0 \cup \tilde{b}, 0)]) \\
= \ & 0
\end{aligned}
$$

Similarly, we can show the product does not depend on representatives of β either.

Theorem 6.9. Product formula of Deligne cohomology II
For any Deligne class $\alpha \in H_{\mathcal{D}}^k(X, \mathbb{Z}(p))$ and $\beta \in H_{\mathcal{D}}^l(X, \mathbb{Z}(q))$, there exist Čech-Dolbeault spark representations (a, r) for α and (b, s) for β as above. Let $(\tilde{a}, r)$ and $(\tilde{b}, s)$ be smooth hypersparks which are lifts of (a, r) and (b, s). Then

$$\alpha * \beta = \Pi_{p+q}[(\tilde{a} \cup s + (-1)^k e \cup \tilde{b}, r \cup s)] = [(\pi_{p+q}(\tilde{a} \cup s + (-1)^k e \cup b), r \cup s)] \in H_{\mathcal{D}}^{k+l}(X, \mathbb{Z}(p+q)).$$

Theorem 6.10. *Two product formulas in Theorem 6.5 and Theorem 6.9 are equivalent.*

Proof. The product formula in Theorem 6.5 and 6.9 are based on product formulas of $\hat{\mathbf{H}}_{spark}^*(X, p)$ and $\hat{\mathbf{H}}_{smooth}^*(X, p)$ established in [HL2] and [H1] respectively. Note that $\hat{\mathbf{H}}_{spark}^*(X, p) \cong \hat{\mathbf{H}}_{smooth}^*(X, p)$ and the ring structures on them are compatible [H1, Theorem 4.5]. Hence, product formulas in Theorem 6.5 and 6.9 are equivalent as well. $\square$

Our product formula is quite explicit compared with the product in [B] which is defined on the sheaf level. Now we verify that these products are equivalent.

Theorem 6.11. *The products Theorem 6.5 and 6.9 are equivalent to Beilinson's product.*

Proof. It is sufficient to show that the product formula in Theorem 6.9 is the same as Beilinson's product which is induced from the cup product on the sheaf level.

The outline of the proof is following: First, we construct an explicit isomorphism between $H_{\mathcal{D}}^k(X, \mathbb{Z}(p))$ and $\ker \delta_1 : \hat{\mathbf{H}}_{smooth}^{k-1}(X, p) \to \mathcal{Z}_{\mathbb{Z}}^k(X, p)$; Then, we calculate the product induced by

$$\cup : \mathbb{Z}_{\mathcal{D}}(p) \otimes \mathbb{Z}_{\mathcal{D}}(q) \to \mathbb{Z}_{\mathcal{D}}(p+q)$$

using Čech resolution; Finally, we calculate the product via smooth hypersparks defined earlier in this section, and compare these two products.

Step 1: Fix a good cover $\{\mathcal{U}\}$ of X and take Čech resolution for the complex of sheaves $\mathbb{Z}_{\mathcal{D}}(p) \longrightarrow \mathcal{C}^*(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}(p))$.

Then

$$H_{\mathcal{D}}^q(X, \mathbb{Z}(p)) \equiv \mathbb{H}^q(\mathbb{Z}_{\mathcal{D}}(p)) \cong \mathbb{H}^q(Tot(\mathcal{C}^*(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}(p)))) \cong H^q(Tot(C^*(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}(p))))$$

where $C^*(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}(p))$ are the groups of global sections of sheaves $\mathcal{C}^*(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}(p))$ and look like the following double complex.

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta \\
C^k(\mathcal{U},\mathbb{Z}) \xrightarrow{(-1)^k i} & C^k(\mathcal{U},\Omega^0) \xrightarrow{(-1)^k\partial} & C^k(\mathcal{U},\Omega^1) \xrightarrow{(-1)^k\partial} & C^k(\mathcal{U},\Omega^2) \xrightarrow{(-1)^k\partial} \cdots \xrightarrow{(-1)^k\partial} & C^k(\mathcal{U},\Omega^{p-1}) \\
\uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta \\
C^2(\mathcal{U},\mathbb{Z}) \xrightarrow{i} & C^2(\mathcal{U},\Omega^0) \xrightarrow{\partial} & C^2(\mathcal{U},\Omega^1) \xrightarrow{\partial} & C^2(\mathcal{U},\Omega^2) \xrightarrow{\partial} \cdots \xrightarrow{\partial} & C^2(\mathcal{U},\Omega^{p-1}) \\
\uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta \\
C^1(\mathcal{U},\mathbb{Z}) \xrightarrow{-i} & C^1(\mathcal{U},\Omega^0) \xrightarrow{-\partial} & C^1(\mathcal{U},\Omega^1) \xrightarrow{-\partial} & C^1(\mathcal{U},\Omega^2) \xrightarrow{-\partial} \cdots \xrightarrow{-\partial} & C^1(\mathcal{U},\Omega^{p-1}) \\
\uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta \\
C^0(\mathcal{U},\mathbb{Z}) \xrightarrow{i} & C^0(\mathcal{U},\Omega^0) \xrightarrow{\partial} & C^0(\mathcal{U},\Omega^1) \xrightarrow{\partial} & C^0(\mathcal{U},\Omega^2) \xrightarrow{\partial} \cdots \xrightarrow{\partial} & C^0(\mathcal{U},\Omega^{p-1})
\end{array}
$$

Let $M_p^* \equiv Tot(C^*(\mathcal{U},\mathbb{Z}_{\mathcal{D}}(p)))$ denote the total complex of the double complex $C^*(\mathcal{U},\mathbb{Z}_{\mathcal{D}}(p))$ with differential

$$
\Delta_p(a) = \begin{cases}
(\delta + (-1)^r i)(a), & \text{when} \quad a \in C^r(\mathcal{U},\mathbb{Z}); \\
(\delta + (-1)^r \partial)(a), & \text{when} \quad a \in C^r(\mathcal{U},\Omega^j), \quad j < p-1; \\
\delta a, & \text{when} \quad a \in C^r(\mathcal{U},\Omega^{p-1}).
\end{cases}
$$

Now we construct a map

$$
\varphi_p : H^*(M_p^*) \cong H_{\mathcal{D}}^*(X,\mathbb{Z}(p)) \longrightarrow \ker\delta_1 \subset \hat{\mathbf{H}}_{smooth}^{*-1}(X,p).
$$

Assume that a cycle $\tilde{a} \in M_p^k$ represents a Deligne class in $H_{\mathcal{D}}^k(X,\mathbb{Z}(p))$, and

$$
\tilde{a} = r + a = r + \sum_{i+j=k-1, j<p} a^{i,j}
$$

where $r \in C^k(\mathcal{U},\mathbb{Z})$ and $a^{i,j} \in C^i(\mathcal{U},\Omega^j)$.

Note that $a^{i,j} \in C^i(\mathcal{U},\Omega^j) \subset C^i(\mathcal{U},\mathcal{E}^{j,0}) \subset C^i(\mathcal{U},\mathcal{E}_p^j)$, so $a \in \bigoplus_{i+j=k-1} C^i(\mathcal{U},\mathcal{E}_p^j)$. And it is easy to see

$$
\Delta_p \tilde{a} = 0 \Leftrightarrow D_p a + (-1)^k r = 0 \text{ and } \delta r = 0,
$$

where D_p is the differential of the total complex of double complex $\bigoplus_{r+s=*} C^r(\mathcal{U},\mathcal{E}_p^s)$ defined in Section 4. Hence, $\varphi_p : \tilde{a} \to (a,(-1)^k r)$ gives a map from cycles to smooth hypersparks. Moreover, assume $\tilde{a}$ and $\tilde{a}'$ represent the same Deligne class, i.e. $\tilde{a} - \tilde{a}' = \Delta_p \tilde{b}$ is a boundary, where $\tilde{b} = s + \Sigma_{i+j=k-2, j<p} b^{i,j}$ for $s \in C^{k-1}(\mathcal{U},\mathbb{Z})$ and $b^{i,j} \in C^i(\mathcal{U},\Omega^j)$. Then

$$
a - a' + r - r' = \tilde{a} - \tilde{a}' = \Delta_p \tilde{b} = \delta s + (-1)^{k-1} i(s) + D_p b
$$

implies

$$
a - a' = (-1)^{k-1} s + D_p b \text{ and } (-1)^k r - (-1)^k r' = -(-1)^{k-1}\delta s,
$$

i.e. $(a,(-1)^k r)$ and $(a',(-1)^k r')$ represent the same spark class. So the map (also denoted by φ_p)

$$
\varphi_p : H_{\mathcal{D}}^k(X,\mathbb{Z}(p)) \to \hat{\mathbf{H}}_{smooth}^{k-1}(X,p)
$$

which maps a Deligne class $[\tilde{a}]$ to a smooth hyperspark class $[(a,(-1)^k r)]$ is well-defined. $\varphi_p([\tilde{a}]) = [(a,(-1)^k r)]$ satisfies the spark equation $D_p a + (-1)^k r = 0$, so we have $Im\varphi_p \subset \ker\delta_1$. We have known $\varphi_p : H_{\mathcal{D}}^k(X,\mathbb{Z}(p)) \to \ker\delta_1$ is an isomorphism from Lemma 6.8.

Step 2: The product formula for Deligne classes is induced by the cup product
$$\cup : \mathbb{Z}_{\mathcal{D}}(p) \otimes \mathbb{Z}_{\mathcal{D}}(q) \to \mathbb{Z}_{\mathcal{D}}(p+q)$$
with the formula
$$x \cup y = \begin{cases} x \cdot y & \text{if } \deg x = 0; \\ x \wedge dy & \text{if } \deg x > 0 \text{ and } \deg y = q; \\ 0 & \text{otherwise.} \end{cases}$$

In the appendix of [H1], we showed the explicit product formula on Čech cycles.

Assume
$$\alpha \in H_{\mathcal{D}}^k(X, \mathbb{Z}(p)) \quad \text{and} \quad \beta \in H_{\mathcal{D}}^l(X, \mathbb{Z}(q)),$$
and let
$$\tilde{a} = r + a = r + \sum_{i+j=k-1, j<p} a^{i,j} \in M_p^k \text{ be a representative of } \alpha$$
and
$$\tilde{b} = s + b = s + \sum_{i+j=l-1, i<q} b^{i,j} \in M_q^l \text{ be a representative of } \beta$$
where
$$r \in C^k(\mathcal{U}, \mathbb{Z}), \quad a^{i,j} \in C^i(\mathcal{U}, \Omega^j),$$
and
$$s \in C^l(\mathcal{U}, \mathbb{Z}), \quad b^{i,j} \in C^i(\mathcal{U}, \Omega^j).$$

By [H1, Theorem 7.1], we calculate
$$\begin{aligned} \alpha \cup \beta &= [\tilde{a} \cup \tilde{b}] \\ &= [rs + \Sigma_{i+j=l-1, j<q}(-1)^{0 \cdot i} r \cdot b^{i,j} + \Sigma_{i+j=k-1, j<p}(-1)^{j \cdot (l-q)} a^{i,j} \wedge \partial b^{l-q, q-1}] \\ &= [r \cup \tilde{b} + a \cup \partial b^{l-q, q-1}] \end{aligned}$$

Step 3: Let us calculate the product of α and β by the formula in Theorem 6.9.

$\varphi_p(\tilde{a}) = (a, (-1)^k r)$ and $\varphi_q(\tilde{b}) = (b, (-1)^l s)$ are two smooth hypersparks which represent Deligne classes α and β respectively. The spark equations associated to them are:
$$D_p a = 0 - (-1)^k r, \quad \delta(-1)^k r = 0$$
and
$$D_p b = 0 - (-1)^l s, \quad \delta(-1)^l s = 0.$$

Because of the surjectivity of the map $\Pi_p : \hat{\mathbf{H}}^*_{smooth}(X) \to \hat{\mathbf{H}}^*_{smooth}(X, p)$, there exist
$$(A, (-1)^k r) \in \bigoplus_{i+j=k-1} C^i(\mathcal{U}, \mathcal{E}^j) \oplus C^k(\mathcal{U}, \mathbb{Z}) \text{ with } \Pi_p[(A, (-1)^k r)] = [(a, (-1)^k r)] = \alpha,$$
and
$$(B, (-1)^l s) \in \bigoplus_{i+j=l-1} C^i(\mathcal{U}, \mathcal{E}^j) \oplus C^l(\mathcal{U}, \mathbb{Z}) \text{ with } \Pi_q[(B, (-1)^l s)] = [(b, (-1)^l s)] = \beta.$$

Assume the spark equations for A and B are $DA = e - (-1)^k r$, $DB = f - (-1)^l s$, then $\pi_p A = a$, $\pi_p e = 0$ and $\pi_q B = b$, $\pi_q f = 0$.

By the product formula in [H1],
$$[(A, r)] * [(B, s)] = [A \cup f + (-1)^k(-1)^k r \cup B, (-1)^k r \cup (-1)^l s] = [A \cup f + r \cup B, (-1)^{k+l} r \cup s].$$

We have
$$D(A \cup f + r \cup B) = e \wedge f - (-1)^{k+l} r \cup s \text{ and } \pi_{p+q}(e \wedge f) = 0,$$

so we get $\Pi_{p+q}[(A \cup f + r \cup B, (-1)^{k+l} r \cup s)] \in H_{\mathcal{D}}^{k+l}(X, \mathbb{Z}(p+q))$, and define it as $\alpha * \beta$.

We compare two results under isomorphism $\varphi_{p+q} : H_{\mathcal{D}}^k(X, \mathbb{Z}(p+q)) \to \ker \delta_1$.

The following Lemma shows that $\varphi_{p+q}(r \cup \tilde{b} + a \cup \partial b^{l-q,q-1}) = (r \cup b + a \cup \partial b^{l-q,q-1}, (-1)^{k+l} r \cup s)$ and $(\pi_{p+q}(A \cup f + r \cup B), (-1)^{k+l} r \cup s)$ represent the same class. $\qquad \square$

Lemma 6.12. $r \cup b + a \cup \partial b^{l-q,q-1} = \pi_{p+q}(A \cup f + r \cup B) + (-1)^k D_{p+q}(a \cup (B - b))$.

Proof. Compare

$$Db = \partial b^{l-q,q-1} + D_p b = \partial b^{l-q,q-1} - (-1)^l s$$

and

$$DB = f - (-1)^l s,$$

we have

$$\partial b^{l-q,q-1} = f - D(B - b).$$

Right hand side
$$
\begin{aligned}
&= \pi_{p+q}(A \cup f + r \cup B) + (-1)^k D_{p+q}(a \cup (B - b)) \\
&= \pi_{p+q}(A \cup f) + \pi_{p+q}(r \cup B) + (-1)^k \pi_{p+q} D(a \cup (B - b)) \\
&= \pi_{p+q}(a \cup f) + \pi_{p+q}(r \cup B) + (-1)^k \pi_{p+q}(Da \cup (B - b) + (-1)^{k-1} a \cup D(B - b)) \\
&= \pi_{p+q}(a \cup f) + \pi_{p+q}(r \cup B) + (-1)^k \pi_{p+q}(D_p a \cup (B - b)) - \pi_{p+q}(a \cup D(B - b)) \\
&= \pi_{p+q}(a \cup f) + \pi_{p+q}(r \cup B) + (-1)^k \pi_{p+q}(-(-1)^k r \cup (B - b)) - \pi_{p+q}(a \cup D(B - b)) \\
&= \pi_{p+q}(a \cup f) + \pi_{p+q}(r \cup B) - \pi_{p+q}(r(B - b)) - \pi_{p+q}(a \cup D(B - b)) \\
&= \pi_{p+q}(a \cup f - a \cup D(B - b)) + \pi_{p+q}(r \cup b) \\
&= \pi_{p+q}(a \cup \partial b^{l-q,q-1}) + \pi_{p+q}(r \cup b) \\
&= a \cup \partial b^{l-q,q-1} + r \cup b \\
&= \text{Left hand side.}
\end{aligned}
$$

$\qquad \square$

7. Application to Algebraic Cycles

We begin this section by observing that, from the viewpoint of spark theory, it is trivial that every analytic subvariety of a complex manifold represents a Deligne cohomology class. Furthermore, when two cycles intersect properly, their intersection represents the product of the Deligne classes they represent. We shall then give a proof of the rational invariance of these Deligne classes in the algebraic setting, thereby giving the well known ring homomorphism

$$\psi : CH^*(X) \to H_{\mathcal{D}}^{2*}(X, \mathbb{Z}(*)).$$

Let V be a subvariety of complex manifold X with codimension p. Then integration over the regular part of V

$$[V](\alpha) \equiv \int_{\text{Reg } V} \alpha, \quad \forall \text{ smooth form } \alpha \text{ with compact support}$$

defines a degree (p, p) current $[V]$ on X. Moreover, $[V]$ is rectifiable [Har], hence $[V] \in \mathcal{IF}^{2p}(X)$. It is easy to see V represents a Deligne class.

Proposition 7.1. $(0, [V])$ *represents a spark class in* $\hat{\mathbf{H}}^{2p-1}(X, p)$. *Moreover, this class belongs to* $H_{\mathcal{D}}^{2p}(X, \mathbb{Z}(p)) = \ker \delta_1 \subset \hat{\mathbf{H}}^{2p-1}(X, p)$.

Proof. Since $[V]$ is of type (p,p), we have $\Psi_p([V]) = 0$ and $(0, [V])$ satisfies the spark equation $d0 = 0 - \Psi_p([V])$. $\qquad \square$

Proposition 7.2. *Let V, W be two subvarieties which intersect properly. Then*
$$[(0, [V])] * [(0, [W])] = [(0, [V \cap W])].$$

Proof. Let V, W be two subvarieties in X with codimension p and q respectively. Let r and s denote currents $[V]$ and $[W]$, then $r \wedge s = [V \cap W]$. Now we calculate the product of two Deligne classes $[(0, r)]$ and $[(0, s)]$. First, fix a lift of $(0, r)$, say (a, r) with spark equation $da = e - r$ and a lift of $(0, s)$, (b, s) with $db = f - s$. Note that $\pi_p(a) = 0$, $\pi_p(e) = 0$ and $\pi_q(b) = 0$, $\pi_q(f) = 0$. By product formula, $[(a, r)][(b, s)] = [(a \wedge f + r \wedge b, r \wedge s)]$. Since $\pi_{p+q}(a \wedge f + r \wedge b) = 0$, we have
$$[(0, [V])] * [(0, [W])] = \Pi_{p+q}([(a \wedge f + r \wedge b, r \wedge s)]) = [(0, r \wedge s)] = [(0, [V \cap W])].$$
$\qquad \square$

Proposition 7.3. *If X is an algebraic manifold of dimension n and V is an algebraic cycle which is rationally equivalent to zero, then V represents zero Deligne class.*

Proof. Assume V is an algebraic cycle with dimension k and codimension p. If V is rationally equivalent to zero, in particular, V represents zero homology class, then $V = dS$ for some rectifiable current with degree $2p - 1$ (and real dimension $2k + 1$). Hence $(0, V)$ is equivalent to $(\pi_p(S), 0)$ as sparks of level p. $(\pi_p(S), 0)$ represents zero class if and only of
$$\pi_p(S) = d_p A + \Psi_p R \text{ for some current } A \in \mathcal{D}'^{2p-2}(X, p) \text{ and closed current } R \in \mathcal{IF}^{2p-1}(X),$$
i.e.
$$[\pi_p(S)] = 0 \in H^{2p-1}(X, \mathbb{C})/F^p H^{2p-1}(X, \mathbb{C}) \oplus H^{2p-1}(X, \mathbb{Z}) \equiv \mathcal{J}^p,$$
which means the Abel-Jacobi invariant of V is zero. It is well known that the Abel-Jacobi invariant is trivial for a cycle rationally equivalent to zero. So we are done. We give a short and direct proof of this fact now.

If V is rationally equivalent to zero, then there is a cycle $W \subset \mathbb{P}^1 \times X$ of codimension p, such that $V = \pi^{-1}(1) - \pi^{-1}(0)$ where $\pi : W \to \mathbb{P}^1$, the restriction of the projection $pr_1 : \mathbb{P}^1 \times X \to \mathbb{P}^1$, is equidimensional over $\mathbb{P}^1$. Define $V_z = \pi^{-1}(z) - \pi^{-1}(0)$, then we have a map $\mu : \mathbb{P}^1 \to \mathcal{J}^p$ which assigns z the Abel-Jacobi invariant of V_z. We shall show that μ is holomorphic, hence a constant map to zero.

Let us recall the construction of the Abel-Jacobi map briefly. If V is a cycle homologous to zero, then $V = dS$. Integrating over S, $\int_S$ defines a class in
$$H^{2p-1}(X, \mathbb{C})/F^p H^{2p-1}(X, \mathbb{C}) = F^{n-p+1} H^{2n-2p+1}(X, \mathbb{C})^*.$$
If $dS' = V$, then the difference $\int_S - \int_{S'}$ lies in the image of map
$$H_{2n-2p+1}(X, \mathbb{Z}) \to F^{n-p+1} H^{2n-2p+1}(X, \mathbb{C})^*.$$
Therefore, we get the Abel-Jacobi invariant of V defined in
$$\mathcal{J}^p \equiv H^{2p-1}(X, \mathbb{C})/F^p H^{2p-1}(X, \mathbb{C}) \oplus H^{2p-1}(X, \mathbb{Z}).$$

Now we focus on the map μ. Let γ_z be a curve on $\mathbb{P}^1$ connecting 0 and z and $S_z = \pi^{-1}(\gamma_z)$ with $dS_z = V_z$. We want to show that $\mu : z \mapsto \int_{S_z}$ is holomorphic. Note that
$$F^{n-p+1} H^{2n-2p+1}(X, \mathbb{C}) \cong \bigoplus_{\substack{r+s=2n-2p+1 \\ r \geq n-p+1}} \mathcal{H}^{r,s}(X)$$
where $\mathcal{H}^{r,s}(X)$ is the group of harmonic (r, s) forms. So it suffices to show $\mu_\alpha : z \mapsto \int_{S_z} \alpha$ is holomorphic for every $\alpha \in \mathcal{H}^{r,s}(X)$, $r + s = 2n - 2p + 1$, $r \geq n - p + 1$.

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Let ν be a vector field in a small neighborhood U of z in $\mathbb{P}^1$, and $\tilde{\nu}$ be a lift of ν in $U \times X$. If ν is of type $(0,1)$, we have

$$\nu \int_{S_z} \alpha = \int_{\pi^{-1}(z)} \tilde{\nu} \lrcorner \alpha = 0$$

for any $\alpha \in \mathcal{H}^{r,s}(X)$, $r + s = 2n - 2p + 1$, $r \geq n - p + 1$. The last equality follows from the fact $\tilde{\nu} \lrcorner \alpha$ has no component of type $(n - p, n - p)$.

$\square$

By the last propositions and Chow's moving lemma, it is easy to see

Theorem 7.4. *The map $V \mapsto [(0, [V])]$ induces a ring homomorphism*

$$\psi : CH^*(X) \to H^{2*}_{\mathcal{D}}(X, \mathbb{Z}(*)).$$

8. Chern Classes for Holomorphic Bundles in Deligne Cohomology

In this section we shall construct Chern classes in Deligne cohomology for holomorphic bundles E over a complex manifold X. These classes have the usual properties and map to the integral Chern classes under the ring homomorphism $H^{2*}_{\mathcal{D}}(X, \mathbb{Z}(*)) \to H^{2*}(X, \mathbb{Z})$.

In their fundamental paper [CS], Cheeger and Simons showed that for a smooth complex vector bundle $E \to X$ with unitary connection ∇, there exist refined Chern classes $\hat{c}_k(E, \nabla) \in \hat{\mathbf{H}}^{2k-1}(X)$ with

$$\delta_1(\hat{c}_k(E, \nabla)) = c_k(\Omega^\nabla) \quad \text{and} \quad \delta_2(\hat{c}_k(E, \nabla)) = c_k(E)$$

where $c_k(E)$ is the kth integral Chern class and $c_k(\Omega^\nabla)$ is the Chern-Weil form representing $c_k(E) \otimes \mathbb{R}$ in the curvature of ∇. Setting $\hat{c}(E) = 1 + \hat{c}_1 + \hat{c}_2 + ...$, they showed

$$\hat{c}(E \oplus E', \nabla \oplus \nabla') = \hat{c}(E, \nabla) * \hat{c}(E', \nabla').$$

When X is a complex manifold, we can take the projections

$$\hat{d}_k(E, \nabla) \equiv \Pi_k(\hat{c}_k(E, \nabla)) \in \hat{\mathbf{H}}^{2k-1}(X, k).$$

By equations above and Proposition 3.8, we have

$$\delta_1(\hat{d}_k(E, \nabla)) = \pi_k(c_k(\Omega^\nabla)) \quad \text{and} \quad \delta_2(\hat{d}_k(E, \nabla)) = c_k(E).$$

Suppose that E is holomorphic and is provided with a hermitian metric h. Let ∇ be the associated canonical hermitian connection. Then $c_k(\Omega^\nabla)$ is of type k, k and we have

$$\delta_1(\hat{d}_k(E, \nabla)) = \pi_k(c_k(\Omega^\nabla)) = 0 \quad \Longrightarrow \quad \hat{d}_k(E, \nabla) \in \ker(\delta_1) = H^{2k}_{\mathcal{D}}(X, \mathbb{Z}(k)).$$

Proposition 8.1. *The class $\hat{d}_k(E, \nabla) \in H^{2k}_{\mathcal{D}}(X, \mathbb{Z}(k))$ defined above is independent of the choice of hermitian metric.*

Proof. Let h_0, h_1 be hermitian metrics on E with canonical connections ∇_0, ∇_1 respectively. Then

$$\hat{c}_k(E, \nabla_1) - \hat{c}_k(E, \nabla_0) = [T]$$

where $[T]$ is the differential character represented by the smooth transgression form

$$T = T(\nabla_1, \nabla_0) \equiv k \int_0^1 C_k(\nabla_1 - \nabla_0, \Omega_t, ..., \Omega_t) dt$$

where $C_k(X_1, ..., X_k)$ is the polarization of the kth elementary symmetric function and where Ω_t is the curvature of the connection $\nabla_t \equiv t\nabla_1 + (1 - t)\nabla_0$. Fix a local holomorphic frame field for E and let H_i be the hermitian matrix representing the metric h_i with respect to this trivialization. Then

$$\nabla_1 - \nabla_0 = \theta_1 - \theta_0 \text{ where } \theta_j \equiv \partial H_j \cdot H_j^{-1}.$$

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In this framing, $\nabla_t = d + \theta_t$ where $\theta_t = t\theta_1 + (1-t)\theta_0$ and so its curvature $\Omega_t = d\theta_t - \theta_t \wedge \theta_t$ only has Hodge components of type $1,1$ and $2,0$. It follows that the Hodge components

$$T^{p,q} = 0 \text{ for } p < q.$$

So we have $\hat{d}_k(E, \nabla_1) - \hat{d}_k(E, \nabla_0) = \Pi_k([T]) = 0$.

$\square$

Remark 8.2. *In the proof of last Proposition, it is easy to see that we can choose any connection compatible to the complex structure ($\nabla^{0,1} = \bar{\partial}$) to define the Chern classes in Deligne cohomology.*

By the proposition above, each holomorphic vector bundle of rank r has a well defined total Chern class in Deligne cohomology

$$\hat{d}(E) = 1 + \hat{d}_1(E) + ... + \hat{d}_r(E) \in \bigoplus_{j=0}^r H_{\mathcal{D}}^{2j}(X, \mathbb{Z}(j)).$$

Denote by $\mathcal{V}^k(X)$ the set of isomorphism classes of holomorphic vector bundles of rank k on X, and by $\mathcal{V}(X) = \coprod_{k \geq 0} \mathcal{V}^k(X)$ the additive monoid under Whitney sum.

Theorem 8.3. *On any complex manifold there is a natural transformation of functors*

$$\hat{d} : \mathcal{V}(X) \to \bigoplus_j H_{\mathcal{D}}^{2j}(X, \mathbb{Z}(j))$$

with the property that:

  (1) $\hat{d}(E \oplus F) = \hat{d}(E) * \hat{d}(F)$,
  (2) $\hat{d} : \mathcal{V}^1(X) \to 1 + H_{\mathcal{D}}^2(X, 1)$ *is an isomorphism,*
  (3) *under the natural map* $\kappa : H_{\mathcal{D}}^{2j}(X, \mathbb{Z}(j)) \to H^{2j}(X, \mathbb{Z})$, $\kappa \circ \hat{d} = c$ *(the total integral Chern class).*

Proof. (1)Suppose E and F are holomorphic bundles with hermitian connections ∇ and ∇', then we have $\hat{d}(E) = 1 + \hat{d}_1(E) + \hat{d}_2(E) + \cdots = 1 + \Pi_1(\hat{c}_1(E, \nabla)) + \Pi_2(\hat{c}_2(E, \nabla)) + \cdots$ and similarly $\hat{d}(F) = 1 + \Pi_1(\hat{c}_1(F, \nabla')) + \Pi_2(\hat{c}_2(F, \nabla')) + \cdots$.

Since

$$\hat{c}(E \oplus F, \nabla \oplus \nabla') = \hat{c}(E, \nabla) * \hat{c}(F, \nabla'),$$

we have

$$
\begin{aligned}
\hat{d}_k(E \oplus F) &= \Pi_k(\hat{c}_k(E \oplus F, \nabla \oplus \nabla')) \\
&= \Pi_k(\sum_{i=0}^k \hat{c}_i(E, \nabla) \cdot \hat{c}_{k-i}(F, \nabla')) \\
&= \sum_{i=0}^k \Pi_i(\hat{c}_i(E, \nabla)) \cdot \Pi_{k-i}(\hat{c}_{k-i}(F, \nabla')) \\
&= \sum_{i=0}^k \hat{d}_i(E) \cdot \hat{d}_{k-i}(F).
\end{aligned}
$$

It is easy to see the second to last equality from our definition of product of Deligne cohomology classes. Recall when we defined the product of two Deligne classes, we first lifted them to two sparks, then did multiplication and projected the product back.

  (2) is true because $H_{\mathcal{D}}^2(X, \mathbb{Z}(1)) \cong H^1(X, \mathcal{O}^*)$.
  (3) follows $\delta_2(\hat{d}_k(E, \nabla)) = c_k(E)$.

$\square$

Following Grothendieck we define the holomorphic K-theory of X to be the quotient

$$K_{\mathrm{hol}}(X) \equiv \mathcal{V}(X)^+ / \sim$$

where $\sim$ is the equivalence relation generated by setting $[E] \sim [E' \oplus E'']$ when there exists a short exact sequence of holomorphic bundles $0 \to E' \to E \to E'' \to 0$. The next theorem tells us the natural transformation $\hat{d}$ defined above descends to a natural transformation

$$\hat{d} : K_{\mathrm{hol}}(X) \to \bigoplus_j H_{\mathcal{D}}^{2j}(X, \mathbb{Z}(j)).$$

Theorem 8.4. *For any short exact sequence of holomorphic vector bundles on X*

$$0 \to E' \to E \to E'' \to 0$$

one has $\hat{d}(E) = \hat{d}(E') * \hat{d}(E'')$.

Proof. We have $E' \oplus E'' \cong E$ as smooth bundles, so we consider them as the same bundle with different holomorphic structures. The purpose is to show these two holomorphic bundles have the same total Chern class valued in Deligne cohomology. The idea of the proof is the following. We fix a hermitian metric on this smooth bundle, choose local holomorphic bases for those two holomorphic structures respectively, and calculate the hermitian connections with respect to them. Then we calculate the smooth transgression form which represents the difference of Cheeger-Simons Chern classes of these two holomorphic bundles, and show that under the projection Π_k, this transgression form represents a zero spark class in $\hat{\mathbf{H}}^{2k-1}(X, k)$. Hence $\hat{d}(E) = \hat{d}(E' \oplus E'') = \hat{d}(E') * \hat{d}(E'')$.

Choose a C^∞-splitting
$$0 \longrightarrow E' \overset{i}{\longrightarrow} E \underset{\sigma}{\overset{\pi}{\rightleftarrows}} E'' \longrightarrow 0.$$

Fix hermitian metrics h_1 and h_2 for E' and E'' respectively, and define a hermitian metric $h = h_1 \oplus h_2$ on E via the smooth isomorphism $(i, \sigma) : E' \oplus E'' \to E$.

Over a small open set $U \subset X$, we choose a local holomorphic basis $\{e_1, e_2, ..., e_m\}$ for E' and a local holomorphic basis $\{e_{m+1}, e_{m+2}, ..., e_{m+n}\}$ for E''. Then choose a local holomorphic basis $\{\tilde{e}_1, \tilde{e}_2, ..., \tilde{e}_m, \tilde{e}_{m+1}, \tilde{e}_{m+2}, ..., \tilde{e}_{m+n}\}$ for E such that $\tilde{e}_i = e_i$ for $1 \le i \le m$ and $\tilde{e}_{m+j}$ is a holomorphic lift of e_{m+j} for $1 \le j \le n$. Assume $g = (g_{ij})$ is the transition matrix for these two bases, i.e. $\tilde{e}_i = \sum_{j=1}^{m+n} g_{ij} e_j$. Then it is easy to know g has the form

$$g = \begin{pmatrix} I_m & 0 \\ A & I_n \end{pmatrix} \quad \text{and} \quad g^{-1} = \begin{pmatrix} I_m & 0 \\ -A & I_n \end{pmatrix}$$

where I_m and I_n is the identity matrices of rank m and n, and A is the nontrivial part of g.

Let H_1 and H_2 be the hermitian matrices representing the metrics h_1 and h_2 with respect to the bases $\{e_1, e_2, ..., e_m\}$ and $\{e_{m+1}, e_{m+2}, ..., e_{m+n}\}$. Let H and $\tilde{H}$ be the hermitian matrices representing the metric h with respect to bases $\{e_i\}_{i=1}^{m+n}$ and $\{\tilde{e}_i\}_{i=1}^{m+n}$. Then we have

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \quad \text{and} \quad \tilde{H} = gHg^*$$

where $g^* = \bar{g}^t$ is the transpose conjugate of g.

Fix the hermitian metric h, we calculate the canonical hermitian connections with respect to two holomorphic structures. For $E' \oplus E''$, the hermitian connection ∇_0 can be written locally as the matrix (w.r.t. the basis $\{e_i\}$)

$$\theta_0 = \partial H \cdot H^{-1}.$$

For E, the hermitian connection ∇_1 can be written locally as the matrix (w.r.t. the basis $\{\tilde{e}_i\}$)

$$\tilde{\theta}_1 = \partial \tilde{H} \cdot \tilde{H}^{-1} = \partial(gHg^*)(gHg^*)^{-1} = \partial g \cdot g^{-1} + g\partial H \cdot H^{-1}g^{-1} + gH\partial g^*(g^*)^{-1}H^{-1}g^{-1}.$$

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We change the basis and write ∇_1 as the matrix with respect to the basis $\{e_i\}$

$$
\begin{aligned}
\theta_1 &= d(g^{-1}) \cdot g + g^{-1}\tilde{\theta}_1 g \\
&= -g^{-1}dg + g^{-1}(\partial g \cdot g^{-1} + g\partial H \cdot H^{-1}g^{-1} + gH\partial g^*(g^*)^{-1}H^{-1}g^{-1})g \\
&= -g^{-1}dg + g^{-1}\partial g + \partial H \cdot H^{-1} + H\partial g^*(g^*)^{-1}H^{-1} \\
&= -g^{-1}\bar{\partial}g + \theta_0 + H\partial g^*(g^*)^{-1}H^{-1} \\
&= \theta_0 + H\partial g^*(g^*)^{-1}H^{-1} - g^{-1}\bar{\partial}g
\end{aligned}
$$

Let $\eta \equiv \theta_1 - \theta_0 = H\partial g^*(g^*)^{-1}H^{-1} - g^{-1}\bar{\partial}g$ and $\eta^{1,0} = H\partial g^*(g^*)^{-1}H^{-1}$, $\eta^{0,1} = -g^{-1}\bar{\partial}g$ be the $(1,0)$ and $(0,1)$ components of η respectively. Then we have

$$
\eta^{1,0} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}\begin{pmatrix} 0 & \partial A^* \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_m & -A^* \\ 0 & I_n \end{pmatrix}\begin{pmatrix} H_1^{-1} & 0 \\ 0 & H_2^{-1} \end{pmatrix} = \begin{pmatrix} 0 & H_1\partial A^* H_2^{-1} \\ 0 & 0 \end{pmatrix}
$$

and

$$
\eta^{0,1} = -\begin{pmatrix} I_m & 0 \\ -A & I_n \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \bar{\partial}A & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ \bar{\partial}A & 0 \end{pmatrix}.
$$

Define a family of connections ∇_t with connection matrices $\theta_t = \theta_0 + t\eta$ for $0 \le t \le 1$. Let $\Omega_t = d\theta_t - \theta_t \wedge \theta_t$ be the curvature of the connection θ_t. It is easy to see

$$
\Omega_t^{0,2} = t\bar{\partial}\eta^{0,1} - t^2(\eta^{0,1} \wedge \eta^{0,1}) = 0
$$

and

$$
\Omega_t^{1,1} = \bar{\partial}\theta_0 + t(\bar{\partial}\eta^{1,0} + \partial\eta^{0,1} - \theta_0 \wedge \eta^{0,1} - \eta^{0,1} \wedge \theta_0) - t^2(\eta^{1,0} \wedge \eta^{0,1} + \eta^{0,1} \wedge \eta^{1,0}).
$$

Note that

$$
\eta^{0,1} \wedge \eta^{0,1} = \begin{pmatrix} 0 & 0 \\ -\bar{\partial}A & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 \\ -\bar{\partial}A & 0 \end{pmatrix} = 0.
$$

We will use this trick again in the later calculation.

Suppose that Φ is an symmetric invariant k-multilinear function on the Lie algebra $\mathfrak{gl}_{m+n}(\mathbb{C})$. Then the two connections ∇_0 and ∇_1 on E give rise to two Cheeger-Simons differential characters $\hat{\Phi}_0$ and $\hat{\Phi}_1$, and the difference

$$
\hat{\Phi}_0 - \hat{\Phi}_1 = [T_\Phi]
$$

where $[T_\Phi]$ is the character associated to the smooth form

$$
T_\Phi = k \int_0^1 \Phi(\eta, \Omega_t, \Omega_t, ..., \Omega_t)dt.
$$

Our goal is to show that $\Pi_k([T_\Phi]) = [\pi_k(T_\Phi)]$ represents a zero spark class. So it suffices to show $\pi_k(T_\Phi)$ is a d_k-exact form. In fact, we shall show $\pi_k(T_\Phi)$ is a form of pure type $(k-1,k)$ and equals $\bar{\partial}S = d_k S$ for some $(k-1,k-1)$ form S.

Lemma 8.5. $T_\Phi^{i,2k-1-i} = 0$ for $i < k-1$, i.e. $\pi_k(T_\Phi) = T_\Phi^{k-1,k}$, where $T_\Phi^{i,2k-1-i}$ is the $(i, 2k-1-i)$ Hodge component of T_Φ.

Proof. Note that we have $\Omega_t^{0,2} = 0$, i.e. Ω_t is of type $(1,1)$ and $(2,0)$. Hence it is easy to see $T_\Phi^{i,2k-1-i} = 0$ for $i < k-1$ from the expression $T_\Phi = k\int_0^1 \Phi(\eta, \Omega_t, \Omega_t, ..., \Omega_t)dt$. $\square$

In order to show T_Φ is $\bar{\partial}$-exact for general Φ, we first show T_{Ψ_k} is $\bar{\partial}$-exact for $\Psi_k(A_1, A_2, ..., A_k) = tr(A_1 \cdot A_2 \cdot ... \cdot A_k)$.

Lemma 8.6. Let $\Psi_k(A_1, A_2, ..., A_k) = tr(A_1 \cdot A_2 \cdot ... \cdot A_k)$ and $T = T_{\Psi_k} = k\int_0^1 tr(\eta \wedge (\Omega_t)^{k-1})dt$. Then $T^{k-1,k}$ is $\bar{\partial}$-exact. Explicitly, $T^{0,1} = 0$ when $k = 1$, and for $k \ge 2$,

$$
T^{k-1,k} = k\int_0^1 tr(\eta^{0,1} \wedge (\Omega_t^{1,1})^{k-1})dt = k\bar{\partial}\int_0^1 -tr(\eta^{0,1} \wedge \eta^{1,0} \wedge (\Omega_t^{1,1})^{k-2}) \cdot t\, dt.
$$

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Proof. When $k = 1$, $T^{0,1} = \int_0^1 tr(\eta^{0,1})dt = 0$ since $\eta^{0,1}$ has the form

$$\begin{pmatrix} 0 & 0 \\ -\bar{\partial}A & 0 \end{pmatrix}.$$

When $k \geq 2$, it is easy to see

$$T^{k-1,k} = k \int_0^1 tr(\eta^{0,1} \wedge (\Omega_t^{1,1})^{k-1})dt$$

by comparing Hodge components on both sides. So it suffices to show the identity

$$tr(\eta^{0,1} \wedge (\Omega_t^{1,1})^{k-1}) = -\bar{\partial}tr(\eta^{0,1} \wedge \eta^{1,0} \wedge (\Omega_t^{1,1})^{k-2}) \cdot t.$$

First, we introduce some basic identities. We know that in our theory, Chern classes in Deligne cohomology are independent of the choice of hermitian metric, and the question above is local. So we fix local bases and choose hermitian metrics h_1 and h_2 such that $H_1 = I_m$ and $H_2 = I_n$ locally. Now we have

$$\eta^{1,0} = \begin{pmatrix} 0 & H_1 \partial A^* H_2^{-1} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \partial A^* \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \eta^{0,1} = \begin{pmatrix} 0 & 0 \\ -\bar{\partial}A & 0 \end{pmatrix}$$

$$\Omega_t^{1,1} = t(\bar{\partial}\eta^{1,0} + \partial\eta^{0,1}) - t^2(\eta^{1,0} \wedge \eta^{0,1} + \eta^{0,1} \wedge \eta^{1,0}) = td\eta - t^2\eta \wedge \eta.$$

Note in the equation above, we use the fact $\eta^{1,0}$ is ∂-exact, $\eta^{0,1}$ is $\bar{\partial}$-exact, and $\eta^{1,0} \wedge \eta^{1,0} = 0$, $\eta^{0,1} \wedge \eta^{0,1} = 0$ by matrix multiplication.

By calculation, we have

$$[\Omega_t^{1,1}, \eta] \equiv \Omega_t^{1,1} \wedge \eta - \eta \wedge \Omega_t^{1,1} = td(\eta \wedge \eta).$$

Take $(2,1)$ and $(1,2)$ components respectively, we have

$$[\Omega_t^{1,1}, \eta^{1,0}] = t\partial(\eta \wedge \eta) \quad \text{and} \quad [\Omega_t^{1,1}, \eta^{0,1}] = t\bar{\partial}(\eta \wedge \eta).$$

The next observation is

$$\bar{\partial}\Omega_t^{1,1} = -t^2\bar{\partial}(\eta \wedge \eta) = -t[\Omega_t^{1,1}, \eta^{0,1}].$$

Using identities above, it is easy to conclude

$$\bar{\partial}((\Omega_t^{1,1})^n) = -t[(\Omega_t^{1,1})^n, \eta^{0,1}].$$

Now we are ready to calculate.

$$\begin{aligned}
& -\bar{\partial}tr(\eta^{0,1} \wedge \eta^{1,0} \wedge (\Omega_t^{1,1})^n) \cdot t \\
=\ & -t \cdot tr(\bar{\partial}(\eta^{0,1} \wedge \eta^{1,0} \wedge (\Omega_t^{1,1})^n)) \\
=\ & -t \cdot tr(-\eta^{0,1} \wedge \bar{\partial}\eta^{1,0} \wedge (\Omega_t^{1,1})^n + \eta^{0,1} \wedge \eta^{1,0} \wedge \bar{\partial}(\Omega_t^{1,1})^n) \\
=\ & -t \cdot tr(-\eta^{0,1} \wedge \bar{\partial}\eta^{1,0} \wedge (\Omega_t^{1,1})^n + \eta^{0,1} \wedge \eta^{1,0} \wedge (-t)[(\Omega_t^{1,1})^n, \eta^{0,1}]) \\
=\ & t \cdot tr(\eta^{0,1} \wedge \bar{\partial}\eta^{1,0} \wedge (\Omega_t^{1,1})^n + t\eta^{0,1} \wedge \eta^{1,0} \wedge (\Omega_t^{1,1})^n \wedge \eta^{0,1} - t\eta^{0,1} \wedge \eta^{1,0} \wedge \eta^{0,1} \wedge (\Omega_t^{1,1})^n) \\
=\ & t \cdot tr(\eta^{0,1} \wedge \bar{\partial}\eta^{1,0} \wedge (\Omega_t^{1,1})^n - t\eta^{0,1} \wedge \eta^{1,0} \wedge \eta^{0,1} \wedge (\Omega_t^{1,1})^n) + t^2 \cdot tr(\eta^{0,1} \wedge \eta^{1,0} \wedge (\Omega_t^{1,1})^n \wedge \eta^{0,1}) \\
=\ & tr(\eta^{0,1} \wedge (t\bar{\partial}\eta^{1,0} - t^2\eta^{1,0} \wedge \eta^{0,1}) \wedge (\Omega_t^{1,1})^n) + t^2 \cdot tr(\eta^{0,1} \wedge \eta^{0,1} \wedge \eta^{1,0} \wedge (\Omega_t^{1,1})^n) \\
\overset{*}{=}\ & tr(\eta^{0,1} \wedge \Omega_t^{1,1} \wedge (\Omega_t^{1,1})^n) + 0 \\
=\ & tr(\eta^{0,1} \wedge (\Omega_t^{1,1})^{n+1})
\end{aligned}$$

Put $n = k - 2$, we are done.

Note that in the second to last equality, we use the trick $\eta^{0,1} \wedge \eta^{0,1} = 0$ several times.

$\square$

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Recall in the Chern-Weil theory, the kth Chern character of a vector bundle is represented by the form $\frac{1}{k!}\Psi_k(\Omega,\Omega,...,\Omega) = \frac{1}{k!}tr(\Omega^k)$ where Ω is the curvature of any connection. Any symmetric invariant k-multilinear function Φ on the Lie algebra $\mathfrak{gl}_{m+n}(\mathbb{C})$ is generated by Ψ_1, Ψ_2, ... Ψ_k, i.e. we have

$$\Phi = \sum_{n=1}^{k} \sum_{\substack{i_1+\cdots+i_n=k \\ i_1>0,\cdots,i_n>0}} a_{i_1 i_2 \cdots i_n} \Psi_{i_1} \otimes \Psi_{i_2} \otimes ... \otimes \Psi_{i_n}.$$

Hence, $T_\Phi = k \int_0^1 \Phi(\eta,\Omega_t,\Omega_t,...,\Omega_t)dt$ where $\Phi(\eta,\Omega_t,\Omega_t,...,\Omega_t)$ is a sum with summands like $\Psi_{i_1}(\eta,\Omega_t,\Omega_t,...,\Omega_t)\Psi_{i_2}(\Omega_t,\Omega_t,...,\Omega_t)...\Psi_{i_n}(\Omega_t,\Omega_t,...,\Omega_t)$. For $j>1$, $\Psi_{i_j}(\Omega_t,\Omega_t,...,\Omega_t)$ is a closed (i_j,i_j) form representing $i_j!$ times the i_jth Chern character. And from last lemma, we know $\Psi_{i_1}(\eta,\Omega_t,\Omega_t,...,\Omega_t)$ has types (i_1-1,i_1) and higher, and its (i_1-1,i_1) component is $\bar\partial$-exact. Therefore, T_Φ is of types $(k-1,k)$ and higher, and $\pi_k(T) = T^{k-1,k} = \bar\partial S^{k-1,k-1}$ for some $(k-1,k-1)$ form $S^{k-1,k-1}$. $\qquad\square$

Remark 8.7. *By the theorems on uniqueness of Chern classes in Deligne cohomology in [B], [EV] and [Br2], our theory on Chern classes is equivalent to all others. In particular, when the setting is algbraic, Chern classes defined above are compatible with Grothendieck-Chern classes under the cycle map $\psi : CH^*(X) \to H_{\mathcal{D}}^{2*}(X,\mathbb{Z}(*))$.*

Cheeger and Simons also defined Chern characters for vector bundles with connections, which are located in rational differential characters $\hat{H}^*(X,\mathbb{R}/\mathbb{Q})$. For holomorphic vector bundles, we can project Chern characters in differential characters to get Chern characters in rational Deligne cohomology $H_{\mathcal{D}}^{2*}(X,\mathbb{Q}(*))$. Define

$$\widehat{dch}_k(E) \equiv \Pi_k(\hat{ch}_k(E,\nabla)) \in H_{\mathcal{D}}^{2k}(X,\mathbb{Q}(k)),$$

where ∇ is the hermitian connection associated to a hermitian metric.

Since $\hat{ch}(E \oplus E', \nabla \oplus \nabla') = \hat{ch}(E,\nabla) + \hat{ch}(E',\nabla')$, we have

Theorem 8.8. *If E and F be two holomorphic vector bundles on complex manifold X, then*

$$\widehat{dch}(E \oplus F) = \widehat{dch}(E) + \widehat{dch}(F).$$

Moreover,

Theorem 8.9. *For any short exact sequence of holomorphic vector bundles on X*

$$0 \to E' \to E \to E'' \to 0$$

one has $\widehat{dch}(E) = \widehat{dch}(E') + \widehat{dch}(E'')$.

9. Bott Vanishing for Holomorphic Foliations

In [Bo], Bott constructed a family of connections on the normal bundle of any smooth foliation of a manifold and established the Bott vanishing theorem which says the characteristic classes of the normal bundle are trivial in all sufficiently high degrees. We shall show an analogue of the Bott vanishing theorem for Chern classes of the normal bundle of a holomorphic foliation.

Suppose that N is the normal bundle to a holomorphic foliation of codimension p on a complex manifold X. Then there are two natural families of connections to consider on N, the family of Bott connections and the family of canonical hermitian connections.

Proposition 9.1. *N is a holomorphic vector bundle on X as above. Let $P(c_1,...,c_q)$ be a polynomial in Chern classes which is of pure degree $2k$ with $k > 2q$. Then the projection image of Cheeger-Simons Chern class $\Pi_k(P(\hat{c}_1,...,\hat{c}_q)) \in \hat{H}^{2k-1}(X,k)$ for Bott connections agrees with the Chern class in Deligne cohomology $P(\hat{d}_1,...,\hat{d}_q)$ for the canonical hermitian connections.*

Proof. In fact, this is a direct corollary of Remark 8.2 since Bott connections are compatible with the holomorphic structure of N.

Let ∇ be a Bott connection, $\tilde{\nabla}$ be the canonical hermitian connection for some hermitian metric and θ, $\tilde{\theta}$ be their connection forms. Notice that both θ, $\tilde{\theta}$ are of type $(1,0)$.

Let $\Phi(X_1, ..., X_k)$ be the symmetric invariant k-multilinear function on the Lie algebra $\mathfrak{gl}_q(\mathbb{C})$ such that $P(\sigma_1(X), ..., \sigma_q(X)) = \Phi(X, ..., X)$ where σ_j is the jth elementary symmetric function of the eigenvalues of X. Then the difference between the Cheeger-Simons Chern class associated to P for the two connections ∇ and $\tilde{\nabla}$ is the character associated to the smooth form

$$T = k \int_0^1 \Phi(\theta - \tilde{\theta}, \Omega_t, ..., \Omega_t) dt$$

where $\Omega_t = d\theta_t - \theta_t \wedge \theta_t$. Since θ_t is of type $(1,0)$ and Ω_t is of type $(1,1)$ and $(2,0)$, we have $T^{p,q} = 0$ for all $p < q$.

Therefore,

$$
\begin{aligned}
& P(\hat{d}_1, ..., \hat{d}_q) - \Pi_k(P(\hat{c}_1(N, \nabla), ..., \hat{c}_q(N, \nabla))) \\
= {} & \Pi_k(P(\hat{c}_1(N, \tilde{\nabla}), ..., \hat{c}_q(N, \tilde{\nabla}))) - \Pi_k(P(\hat{c}_1(N, \nabla), ..., \hat{c}_q(N, \nabla))) \\
= {} & \Pi_k(T) = 0
\end{aligned}
$$

$\square$

Theorem 9.2. *Let N be a holomorphic bundle of rank q on a complex manifold X. If N is (isomorphic to) the normal bundle of a holomorphic foliation of X, then for every polynomial P of pure degree $k > 2q$, the associated refined Chern class satisfies*

$$P(\hat{d}_1(N), ..., \hat{d}_q(N)) \in Im[H^{2k-1}(X, \mathbb{C}^\times) \to H_{\mathcal{D}}^{2k}(X, \mathbb{Z}(k))].$$

Proof. We have the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^{2k-1}(X, \mathbb{C}^\times) & \longrightarrow & \hat{H}^{2k-1}(X) & \longrightarrow & \mathcal{Z}_{\mathbb{Z}}^{2k}(X) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_{\mathcal{D}}^{2k}(X, \mathbb{Z}(k)) & \longrightarrow & \hat{H}^{2k-1}(X, k) & \longrightarrow & \mathcal{Z}_{\mathbb{Z}}^{2k}(X, k) & \longrightarrow & 0
\end{array}
$$

We know $P(\hat{d}_1(N), ..., \hat{d}_q(N)) \in H_{\mathcal{D}}^{2k}(X, \mathbb{Z}(k))$ and $P(\hat{c}_1, ..., \hat{c}_q) \in \hat{H}^{2k-1}(X)$. By last proposition, we know they have the same images in $\hat{H}^{2k-1}(X, k)$. And by Bott vanishing theorem, we have $P(\hat{c}_1, ..., \hat{c}_q) \in H^{2k-1}(X, \mathbb{C}^\times)$. Then we get the conclusion. $\square$

10. Nadel Invariants for Holomorphic Vector Bundles

In his beautiful paper [N], Nadel introduced interesting relative invariants for holomorphic vector bundles. Explicitly, for two holomorphic vector bundles E and F over a complex manifold X and a C^∞ isomorphism $f : E \to F$, Nadel defined invariants $\mathscr{E}^k(E, F, f) \in H^{2k-1}(X, \mathcal{O})$ and $\mathscr{E}^k(E, F) \in H^{2k-1}(X, \mathcal{O})/H^{2k-1}(X, \mathbb{Z})$. He also conjectured that these invariants should coincide with a component of the Abel-Jacobi image of $k!(ch_k(E) - ch_k(F)) \in CH^k(X)$ when the setting is algebraic. In [Be], Berthomieu developed a relative K-theory and gave an affirmative answer to Nadel's conjecture.

In this section, we shall generalize Nadel theory and give a short proof of his conjecture. From our point of view, if E and F are two holomorphic vector bundles whose underlying C^∞ vector bundles are isomorphic, then their usual Chern classes coincide, and the difference of their kth Chern classes in Deligne cohomology $\hat{d}_k(E) - \hat{d}_k(F)$ is located in the intermediate Jacobians $\mathcal{J}^k(X)$. Hence we can define relative invariants for a pair (E, F) directly in $\mathcal{J}^k(X)$. In particular, we shall express the difference of kth Chern character $\widehat{dch}_k(E) - \widehat{dch}_k(F)$ by transgression forms whose $(0, 2k - 1)$

components are exactly the Nadel invariants. This will prove Nadel's conjecture in more general context (not necessarily algebraic).

Let E and F be two holomorphic vector bundles over complex manifold X, and $g : E \to F$ be a C^∞ bundle isomorphism. Fix a hermitian metric h for E and F. Over a small open set $U \subset X$, choose local holomorphic bases $\{e_i\}_{i=1}^r$ and $\{f_i\}_{i=1}^r$ for E and F respectively, and denote also by g the transition matrix of the C^∞ bundle isomorphism with respect to bases $\{e_i\}_{i=1}^r$ and $\{f_i\}_{i=1}^r$. Let H and $\tilde{H}$ be the hermitian matrices representing the metric h with respect to bases $\{e_i\}_{i=1}^r$ and $\{f_i\}_{i=1}^r$. Then we have $\tilde{H} = gHg^*$.

Now we calculate the canonical hermitian connections with respect to two holomorphic structures. For E, the hermitian connection ∇_0 can be written locally as the matrix (w.r.t. the basis $\{e_i\}$)

$$\theta_0 = \partial H \cdot H^{-1}.$$

For F, the hermitian connection ∇_1 can be written locally as the matrix (w.r.t. the basis $\{f_i\}$)

$$\tilde{\theta}_1 = \partial \tilde{H} \cdot \tilde{H}^{-1} = \partial(gHg^*)(gHg^*)^{-1} = \partial g \cdot g^{-1} + g\partial H \cdot H^{-1}g^{-1} + gH\partial g^*(g^*)^{-1}H^{-1}g^{-1}.$$

We change the basis and write ∇_1 as the matrix with respect to the basis $\{e_i\}$

$$
\begin{aligned}
\theta_1 &= d(g^{-1}) \cdot g + g^{-1}\tilde{\theta}_1 g \\
&= -g^{-1}dg + g^{-1}(\partial g \cdot g^{-1} + g\partial H \cdot H^{-1}g^{-1} + gH\partial g^*(g^*)^{-1}H^{-1}g^{-1})g \\
&= -g^{-1}dg + g^{-1}\partial g + \partial H \cdot H^{-1} + H\partial g^*(g^*)^{-1}H^{-1} \\
&= \theta_0 + H\partial g^*(g^*)^{-1}H^{-1} - g^{-1}\bar{\partial}g
\end{aligned}
$$

Let $\eta = \theta_1 - \theta_0 = H\partial g^*(g^*)^{-1}H^{-1} - g^{-1}\bar{\partial}g$ and $\eta^{1,0} = H\partial g^*(g^*)^{-1}H^{-1}$, $\eta^{0,1} = -g^{-1}\bar{\partial}g$ be the $(1,0)$ and $(0,1)$ components of η respectively. Define a family of connections ∇_t with connection matrices $\theta_t = \theta_0 + t\eta$ for $0 \le t \le 1$. Let $\Omega_t = d\theta_t - \theta_t \wedge \theta_t$ be the curvature of the connection θ_t.

$$\Omega_t^{0,2} = t\bar{\partial}\eta^{0,1} - t^2(\eta^{0,1} \wedge \eta^{0,1}) = t\bar{\partial}(-g^{-1}\bar{\partial}g) - t^2(-g^{-1}\bar{\partial}g) \wedge (-g^{-1}\bar{\partial}g) = (t - t^2)g^{-1}\bar{\partial}g \wedge g^{-1}\bar{\partial}g.$$

Suppose that Φ is an symmetric invariant k-multilinear function on the Lie algebra $\mathfrak{gl}_{m+n}(\mathbb{C})$. Then the two connections ∇_0 and ∇_1 on E give rise to two Cheeger-Simons differential characters $\hat{\Phi}_0$ and $\hat{\Phi}_1$, and the difference

$$\hat{\Phi}_0 - \hat{\Phi}_1 = [T_\Phi]$$

where $[T_\Phi]$ is the character associated to the smooth form

$$T_\Phi = k \int_0^1 \Phi(\eta, \Omega_t, \Omega_t, ..., \Omega_t)dt.$$

The difference of Chern classes $\Pi_k([T_\Phi]) = \Pi_k(\hat{\Phi}_0 - \hat{\Phi}_1) \in \mathcal{J}^k$ is a spark class which is represented by the smooth form $\pi_k(T_\Phi)$. In particular, when Φ is the form $\Phi(A_1, A_2, ..., A_k) = tr(A_1 \cdot A_2 \cdot ... \cdot A_k)$ (representing $k!$ times the kth Chern character), we have

$$T_\Phi = k \int_0^1 tr(\eta \wedge (\Omega_t)^{k-1})dt \qquad \text{and} \qquad \pi_k(T_\Phi) = \pi_k(k \int_0^1 tr(\eta \wedge (\Omega_t)^{k-1})dt).$$

The $(0, 2k-1)$ component

$$
\begin{aligned}
\pi_k(T_\Phi)^{0,2k-1} &= T_\Phi^{0,2k-1} \\
&= k \int_0^1 tr(\eta^{0,1} \wedge (\Omega_t^{0,2})^{k-1}) dt \\
&= k \int_0^1 tr(-g^{-1}\bar{\partial}g \wedge ((t-t^2)g^{-1}\bar{\partial}g \wedge g^{-1}\bar{\partial}g)^{k-1}) dt \\
&= k \int_0^1 -t^{k-1}(1-t)^{k-1} tr((g^{-1}\bar{\partial}g)^{2k-1}) dt \\
&= -k \int_0^1 t^{k-1}(1-t)^{k-1} dt \cdot tr((g^{-1}\bar{\partial}g)^{2k-1}) \\
&= \pm \mathscr{E}^k(E, F, g)
\end{aligned}
$$

Definition 10.1. *X is a complex manifold, E and F are two holomorphic vector bundles over X whose underlying C^∞ vector bundle are isomorphic. Define Nadel-type invariants*

$$
\hat{\mathscr{E}}^k(E, F) \equiv k!(\widehat{dch}_k(E) - \widehat{dch}_k(F)) = [\pi_k(k \int_0^1 tr(\eta \wedge (\Omega_t)^{k-1}) dt)] \in \mathcal{J}^k.
$$

Note that we have a natural projection

$$
\pi : \mathcal{J}^k \equiv H^{2k-1}(X, \mathbb{C})/F^k H^{2k-1}(X, \mathbb{C}) \oplus H^{2k-1}(X, \mathbb{Z}) \to H^{2k-1}(X, \mathcal{O})/H^{2k-1}(X, \mathbb{Z}).
$$

By the calculations above, we have

Theorem 10.2. *The kth Nadel invariant $\mathscr{E}^k(E, F)$ is the image of $\hat{\mathscr{E}}^k(E, F)$ under the map $\pi : \mathcal{J}^k \to H^{2k-1}(X, \mathcal{O})/H^{2k-1}(X, \mathbb{Z})$. That is, Nadel's conjecture is true.*

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